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P-STABILITY OF A CLASS OF METHODS
DERIVED FROM THE TAU METHOD

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ABSTRACT

This paper is concerned with the stability analysis of a class of methods derived from the tau method, for certain periodic second order differential equations.
It is proved that the methods are P-stable for every order n if the polynomials chosen in the perturbation term are symmetric in the integration range.

Key words: Second order differential equations, tau method,
P-stability

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1. INTRODUCTION

The tau method was introduced by Lanczos [5] [6 p. 464] in order to obtain polynomial approximants of functions, which are solution of particular differential equations of whatever order.
The class of differential equation to which the tau method can be applied is that which maps polynomials into polynomials (for example, linear differential equations with polynomials coefficients).

The approximating polynomial $y_n(x)$, of degree n, is the exact solution of the differential equation obtained by appending to the right hand side of the given equation a perturbation term of the kind:

$$H_m(x) = \sum_{i=0}^{s} \tau_i \; V_{m-i}(x)$$

$H_m(x)$ is a polynomial, whose degree m depends on the degree n of the approximating polynomial through the differential equation, linear combination of polynomials $V_k(x)$ $(k = m\text{-}s...m)$ suitably chosen, with coefficients τ_i which are determined, together with their number s, by the method.

We emphasize that, for every choice of $V_n(x)$, polynomial basis for the perturbation term, we obtain a class of methods, varing the degree n of the approximating polynomial. The corresponding method will be said of order n.

Let $W_k(x)$ be a polynomial of degree k. Then the polynomials $V_k(x)$ in the perturbation term can be given by:

$$(1.1) \qquad V_k(x) = W_k(x) \qquad\qquad k = m - s \dots m$$

or

$$(1.2) \qquad V_k(x) = x^{k-m+s}\, W_{m-s}(x) \qquad\qquad k = m - s \dots m$$

In the most of cases $W_k(x)$ is an orthogonal polynomial shifted in the integration range. Afterwards the tau method has been extended to linear systems of differential equations with polynomial coefficients [2].

Moreover the tau method has been applied step by step [2] [7] [9] and so its stability properties are of interest.

The results already obtained are the following:

- The tau method applied to systems is A-stable when the $W_k(x)$ are the Chebyshev first kind [1] or Legendre [10] polynomials.
- The tau method applied to a second order differential equation, without first derivative, has finite periodicity intervals when the perturbation term is a linear combination of ultraspherical polynomials [3].

Therefore, when a differential equation of order greater to one has to be solved by the tau method it appears convenient, from the stability point of view, to reduce it to a system of first order differential equations. However, the amount of calculations involved is considerably greater in this latter case than in the first one.

Starting from these considerations and with particular regard to the second order periodic differential equation

$$y''(x) = f(x, y(x))$$

we construct an algorithm which is equivalent to apply the tau method to the system of first order equations obtained by the given equation, but that requires pratically the same computational effort than the tau method applied directly to the equation.

We prove that, if the polynomials $W_k(x)$ are symmetric in the integration range, the method under consideration is P-stable for every order n.

2. THE METHOD

Let us consider the differential problem:

$$(2.1) \quad \begin{cases} A(x)\, y''(x) + B(x)\, y(x) = C(x) & x_0 \leq x \leq x_f \\ y(x_o) = y_0 \\ y'(x_o) = y'_0 \end{cases}$$

where A(x), B(x), C(x) are polynomials of degree respectively p, q, d.

Let us subdivide the integration range into subintervals $[x_i, x_{i+1},]$ of size h, and let us denote by y^1_i , y^2_i the approximate values of $y(x_i)$ and $y'(x_i)$ furnished by the method under consideration.

They are obtained in the following way:

Let n be an integer arbitrarialy fixed and let ${}^iy_n(x)$ be the polynomial of degree n exact solution of the problem:

$$
\begin{cases}
A(x)\ {}^iy''_n(x) + B(x)\ {}^iy_n(x) = C(x) + {}^i\tau_o\ {}^iV'_n(x) + \sum_{j=1}^{s+1} {}^i\tau_j\ {}^iV_{n+j-1}(x) & (2.2) \\
\qquad x_{i-1} \le x \le x_i & \\
{}^iy_n(x_{i-1}) = y^1_{i-1} & (2.2') \\
{}^iy'_n(x_{i-1}) - {}^i\tau_o\ {}^iV_n(x_{i-1}) = y^2_{i-1} & (2.2'')
\end{cases}
$$

where

$$y^1_o = y_o$$

$$y^2_o = y'_o$$

Then we put:

$$y^1_i = {}^iy_n(x_i) \tag{2.3}$$

$$y^2_i = {}^iy'_n(x_i) - {}^i\tau_o\ {}^iV_n(x_i) \tag{2.4}$$

As already said in the introduction n is the order of the method applied and the ${}^iV_k(x)$ are polynomials suitably chosen.

Now the construction of ${}^{i}y_n(x)$ is discussed.

First of all let us introduce the canonical polynomials, which are an useful basis for ${}^{i}y_n(x)$.

They have been completely treated in [8], and we report here only the definitions and results, tailored for the operator under consideration, essential to our purposes.

In order to introduce the canonical polynomials, let us premise some definitions.

DEFINITION 2.1

The m-th generating polynomial of the differential operator (2.1) is:

$$P_m(x) = m(m-1)\ A(x)\, x^{m-2} + B(x)\, x^m$$

$P_m(x)$ is a polynomial of degree at most m+f, where $f = \max\{p-2, q\}$.

Then we can write:

$$(2.5) \qquad P_m(x) = \sum_{k=0}^{m+f} p_k^m\ x^k$$

Let S be the set of integers g such that none linear combination of $P_m(x)$ has degree g. S is a finite set, and s = card (S) in this case is such that [8]:

$$s \leq f+2$$

Let us put:

$$(2.6) \qquad S = \{g_1, g_2 \dots g_s\}$$

and let R_s be the polynomial space spanned by

$$\{x^{g_i}\} \quad i = 1 \dots s$$

Now the canonical polynomials can be defined.

DEFINITION 2.2

The m-th canonical polynomial is the polynomial $Q_m(x)$ such that:

$$(2.7) \qquad A(x)\ Q''_m(x) + B(x)\ Q_m(x) = x^m + R_m(x) \qquad m \in N_o - S$$

where $R_m(x)$, said the m-th residual polynomial, belongs to $\mathcal{R}_s$.

From the (2.6) $R_m(x)$ can be written:

$$(2.8) \qquad R_m(x) = \sum_{j=1}^{s} r^m_j\ x^{g_j} \qquad m \in N_o - S$$

The canonical polynomials and the residuals are connected by the following recursive formulas, useful for their construction ([8]):

$$(2.9) \qquad Q_{m+f}(x) = 1/p^m_{m+f}\ \Big(x^m - \sum_{\substack{i=o \\ i\notin S}}^{m+f-1} p^m_i\ Q_i(x)\Big)$$

$$(2.10) \qquad r^{m+f}_j = 1/p^m_{m+f}\ \Big(-p^m_{g_j} - \sum_{\substack{i=o \\ i\notin S}}^{m+f-1} p^m_i\ r^i_j\Big) \qquad j = 1 \ldots s$$

where p^m_i are the coefficients of the generating polynomial (2.5).

Now the $y_n(x)$ can be constructed. In fact, denoting by c_k and v^n_k the coefficient of x^n respectively in $C(x)$ and ${}^iV_n(x)$, for the linearity of the differential equation and for (2.7) it is obvious that ${}^iy_n(x)$ has the following expression [8]:

$$(2.11) \qquad {}^iy_n(x) = \sum_{\substack{k=o \\ k\notin S}}^{d} c_k\ Q_k(x) + {}^i\tau_o \sum_{\substack{k=1 \\ k\notin S}}^{n} k v^n_k\ Q_{k-1}(x) + \sum_{j=1}^{s+1} {}^i\tau_j \sum_{\substack{k=o \\ k\notin S}}^{n+j-1} v^{n+j-1}_k\ Q_k(x)$$

where s is the cardinal of the set S and the parameter ${}^i\tau_j$ j=0, 1 ... s+1 are solution of the linear system:

$$\sum_{\substack{k=0\\k\neq s}}^{d} c_k Q_k(x_{i-1}) + {}^i\tau_0 \sum_{\substack{k=1\\k\neq s}}^{n} kv^n_k\, Q_{k-1}(x_{i-1}) + \sum_{j=0}^{s+1} {}^i\tau_j \sum_{\substack{k=0\\k\neq s}}^{n+j-1} v^{n+j-1}_k\, Q_k(x_{i-1}) = y^1_{i-1}$$

$$\sum_{\substack{k=0\\k\neq s}}^{d} c_k\, Q_k(x_{i-1}) + {}^i\tau_0 \left(\sum_{\substack{k=1\\k\neq s}}^{n} kv^n_k\, Q'_{k-1}(x_{i-1}) - {}^iV_n(x_{i-1})\right) +$$

$$(2.12) \qquad + \sum_{j=1}^{s+1} {}^i\tau_j \sum_{\substack{k=0\\k\neq s}}^{n+j-1} v^{n+j-1}_k\, Q'_k(x_{i-1}) = y^2_{i-1}$$

$$\sum_{\substack{k=0\\k\neq s}}^{d} c_k\, r_l^k + {}^i\tau_0 \sum_{\substack{k=1\\k\neq s}}^{n} k\, v^n_k\, r_l^{k-1} + \sum_{j=1}^{s+1} {}^i\tau_j \sum_{\substack{k=0\\k\neq s}}^{n+j-1} v^{n+j-1}_k\, r_l^k = c_{g_l} + {}^i\tau_0\, g_l\, v^n_{g_l} + \sum_{j=1}^{s+1} {}^i\tau_j\, v^{n+j-1}_{g_l}$$

$$l = 1 \ldots s$$

obtained by imposing that (2.11) is solution of (2.2).

After that ${}^iy_n(x)$ has been constructed, the approximate values y^1_i, y^2_i, can be easily computed by (2.3) (2.4).

Now the following remarks can be done:

Remark 2.1

The step by step tau method applied to the differential problem (2.1) computes the approximate values of

$y(x_i)$ and $y'(x_i)$, let us say respectively z^1_i, z^2_i, in the following way [9]

Let ${}^iz_n(x)$ be the polynomial of degree n solution of the problem:

$$
(2.12)\quad \begin{cases} A(x)\ {}^{i}z''_n(x) + B(x)\ {}^{i}z_n(x) = C(x) + \sum_{j=o}^{s+1} {}^{i}\tau_j\ {}^{i}V_{n+j-1}(x) \\ {}^{i}z_n(x_{i-1}) = z^1_{i-1} \\ {}^{i}z'_n(x_{i-1}) = z^2_{i-1} \end{cases}
$$

Then:

$$
(2.13)\quad z^1_i = {}^{i}z_n(x_i)
$$

$$
(2.14)\quad z^2_i = {}^{i}z'_n(x_i)
$$

As the construction of ${}^{i}z_n(x)$ is performed analogously to ${}^{i}y_n(x)$ it is obvious that the method introduced in this section has pratically the same computational effort than the tau method applied to (2.1).

Remark 2.2

The differential problem (2.1) can be written as the system:

$$
(2.15)\quad \begin{cases} y'_1(x) - y_2(x) = 0 \\ A(x)\ \ y'_2(x) + B(x)\ \ y_1(x) = C(x) \\ y_1(x_0) = y_0 \\ y_2(x_0) = y'_0 \end{cases}
$$

The step by step tau method applied to (2.15) approximates the values of $y_1(x_i)$, $y_2(x_i)$ by u^1_i, u^2_i, computed in the following way [2].

Let $({}^iu^1_n(x), {}^iu^2_n(x))$ be a vector such that ${}^iu^j_n(x)$ $j = 1,2$ is a polynomial of degree n and:

$$
\begin{cases}
{}^iu^1_n{}'(x) - {}^iu^2_n(x) = \tau_o \, {}^iV_n(x) & (2.16') \\
A(x)\; {}^iu^2_n{}'(x) + B(x)\; {}^iu^1_n(x) = C(x) + \sum_{j=1}^{s+1} {}^i\tau_j \; {}^iV_{n+j-1}(x) & (2.16'') \\
{}^iu^1_n(x_{i-1}) = u^1_{i-1} & (2.16''') \\
{}^iu^2_n(x_{i-1}) = u^2_{i-1} & (2.16^{IV})
\end{cases}
$$

Then

$$u^1_i = {}^iu^1_n(x_i) \quad (2.17)$$

$$u^2_i = {}^iu^2_n(x_i) \quad (2.18)$$

We don't report here the details of the construction of $({}^iu^j_n(x), {}^iu^2_n(x))$ as discussed in [2], since it is behind our purposes.

This method is equivalent to the method previously introduced. In fact, (2.2') can be obtained deriving (2.16') and substituting in (2.16"), and the (2.2"), (2.3), (2.4) follow obviously from (2.16'''), (2.16^{IV}), (2.17), (2.18) taking into account the (2.16').

However, to solve the problem (2.16) as indicated in [2] requires a computational effort substantially greater than to solve (2.2).

Remark 2.3

The method has been introduced here for a second order differential problem without first derivative because of its periodicity properties.

However, it is obvious that it can be trivially generalized for a differential problem of whatever order.

3. STABILITY ANALYSIS

The stability analysis of the method under consideration can be carried out in two ways: proving that it is a D-method [4] or applying it to the second order differential test problem:

$$\begin{cases} y''(x) + \lambda^2 y(x) = 0 & (3.1) \\ y(x_0) = y_0 & (3.1') \\ y'(x_0) = y'_0 & (3.1'') \end{cases}$$

We prefer this latter way, as it is more straightforward.
Without loss of generality, we can consider only the first integration step. Then $x \varepsilon (x_o, x_1)$ and we can drop out the index of the integration step.
For the differential operator (3.1) the set S is empty so the perturbed problem (2.2) is:

$$(3.2)\quad \begin{cases} y''_n(x) + \lambda^2 y_n(x) = \tau_0\ V'_n(x) + \tau_1 V_n(x) & x_0 \le x \le x_1 \\ y_n(x_0) = y_0 \\ y'_n(x_0) - \tau_0 V_n(x_0) = y'_0 \end{cases}$$

We emphasize that the polynomial $V_n(x)$ and the order n are arbitrarialy fixed.

Let $Q_m(x)$ be the m-th canonical polynomial of the differential operator (3.1) and let v^n_k be the coefficient of x^k in $V_n(x)$.

Let us put:

(3.3) $$T_n(x) = \sum_{k=0}^{n} v^n_k \, Q_k(x)$$

For the linearity of (3.1) and since the set S is empty, it is obviously:

(3.4) $$T''_n(x) + \lambda^2 T_n(x) = V_n(x)$$

Therefore $y_n(x)$, solution of (3.2), can be written:

(3.5) $$y_n(x) = \tau_0 \, T'_n(x) + \tau_1 \, T_n(x)$$

and τ_0, τ_1, are solution of the system:

(3.6) $$\begin{cases} \tau_0 \, T'_n(x_0) + \tau_1 \, T_n(x_0) = y_0 \\ \tau_0 \, [T''_n(x_0) - V_n(x_0)] + \tau_1 \, T'_n(x_0) = y'_0 \end{cases}$$

Therefore, solving the (3.6) and using the (3.4) we have:

(3.7) $$\tau_0 = (y_0 \, T'_n(x_0) - y'_0 \, T_n(x_0)) / \Delta$$

(3.8) $$\tau_1 = (y_0 \lambda^2 \, T_n(x_0) + y'_0 \, T'_n(x_0)) / \Delta$$

where:

$$(3.9) \quad \Delta = T'_n(x_0) + \lambda^2 \; T^2_n(x_0)$$

The values $y^1_1 = y_n(x_1)$, $y^2_1 = y_n'(x_1) - \tau_o V_n(x_1)$ are given by:

$$(3.10) \quad y^1_1 = y_0 \; \big(T'_n(x_0)T'_n(x_1) + \lambda^2 \; T_n\,(x_0)T_n(x_1)\big)/\Delta \; +$$

$$+ \; y'_0 \; \big(T'_n(x_0)T_n(x_1) - T_n\,(x_0)T'_n(x_1)\big)/\Delta$$

$$(3.11) \quad y^2_1 = y_0 \; \big(\lambda^2 \, [T_n\,(x_0)T'_n(x_1) - T'_n(x_0)\,T_n(x_1)]\big)/\Delta \; +$$

$$+ \; y'_0 \; \big(\lambda^2 \; T_n\,(x_0)\,T_n(x_1) + T'_n(x_0)\,T'_n(x_1)\big)/\Delta$$

Therefore we can write:

$$\begin{pmatrix} y^1_1 \\ \\ y^2_1 \end{pmatrix} = S \begin{pmatrix} y_0 \\ \\ y'_0 \end{pmatrix}$$

where S is the stability matrix, whose elements are given by:

$$(3.12) \quad s_{11} = \big(\lambda^2 \; T_n\,(x_0)\,T_n(x_1) + T'_n(x_0)\,T'_n(x_1)\big)/\Delta$$

$$(3.13) \quad s_{12} = \big(T'_n(x_0)\,T_n(x_1) - \; T_n\,(x_0)\,T'_n(x_1)\big)/\Delta$$

$$(3.14) \quad s_{21} = \big(T_n(x_0)\,T'_n(x_1) - T'_n(x_0)T_n(x_1)\big)/\Delta$$

$$(3.15) \quad s_{22} = \big(\lambda^2 \; T_n\,(x_0)T_n(x_1) + T'_n(x_0)\,T'_n(x_1)\big)/\Delta$$

where Δ is given by (3.9).

The elements of the stability matrix can be expressed through the polynomial $V_n(x)$. In fact, let $V_n^*(x)$ be the polynomial $V_n(x)$ shifted in [0,1] and let us put $z = (h\lambda)^{-1}$ where $h = x_1 - x_0$ is the integration step. Then the following lemma holds:

Lemma 3.1 -

The elements of the stability matrix have the following expression:

$$(3.16)\quad s_{11} = 1/(\lambda^2\Delta_1)\ \Big[\sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(0) z^{2k} \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(1) z^{2k} + \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(0) z^{2k+1} \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(1) z^{2k+1}\Big]$$

$$(3.17)\quad s_{12} = 1/(\lambda^3\Delta_1)\ \Big[\sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(0) z^{2k+1} \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(1) z^{2k} - \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(0) z^{2k} \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(1) z^{2k+1}\Big]$$

$$(3.18)\quad s_{21} = 1/(\lambda\Delta_1)\ \Big[\sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(0) z^{2k} \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(1) z^{2k+1} - \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(0) z^{2k+1} \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(1) z^{2k}\Big]$$

$$(3.19)\quad s_{22} = 1/(\lambda^2\Delta_1)\ \Big[\sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(0) z^{2k} \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(1) z^{2k} + \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(0) z^{2k+1} \sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(1) z^{2k+1}\Big]$$

where

$$(3.20)\quad \Delta_1 = 1/\lambda^2 \left(\sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k+1)}(0) z^{2k+1}\right)^2$$

$$+\left(\sum_{k=0}^{[n/2]} (-1)^k V_n^{*(2k)}(0) z^{2k}\right)^2$$

Proof:

As it be easily verified:

$$(3.21)\qquad T_n(x) = 1/\lambda^2 \sum_{k=0}^{[n/2]} (-1)^k \left(V_n^{(2k)}(x)\right)/\lambda^{2k}$$

Moreover, it is:

$$V_n^{(j)}(x) = V_n^{*(j)}\left(1/h\ (x-x_0)\right)/h^j \qquad j = 0 \ldots n$$

Then the (3.16) - (3.20) follow by (3.9), (3.12) - (3.15) with elementary algebraic manipulations.

Let us put:

$$(3.22)\qquad \mu_1 = \left(\lambda^2 T_n(x_0)\, T_n(x_1) + T'_n(x_0)\, T'_n(x_1)\right)/\Delta$$

$$(3.23)\qquad \mu_2 = \lambda\left[T'_n(x_0)\, T_n(x_1) - T_n(x_0)\, T'_n(x_1)\right]/\Delta$$

$$(3.24)\quad \mu = \mu_1 + i\mu_2$$

where i is the imaginary unity.

Then the following theorem can be proved.

Theorem 3.1

The eigenvalues of the stability matrix S are μ and $\bar{\mu}$.

Proof:

Let us consider the matrix:

$$P = \begin{pmatrix} 1 & 1 \\ i\lambda & -i\lambda \end{pmatrix}$$

Then:

$$P^{-1} = 1/2 \begin{pmatrix} 1 & -i/\lambda \\ 1 & i/\lambda \end{pmatrix}$$

and with elementary algebraic manipulations we have:

$$P^{-1}\, SP = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}$$

and so the result is proved.

Let us observe that, from (1.1), (1.2) as s=0, and so m=n, we have in both the cases:

$$V_n(x) = W_n(x)$$

Now the P-stability of the methods can be proved, under the hypothesis that $W_n(x)$ is asymmetric in the integration range $[x_0, x_1]$. We emphasize that this hypothesis is satisfied be the most frequently used perturbation term, where the $W_n(x)$ are the Chebyshev (I and II kind), and Legendre polynomials, and also if $W_n(x)$ are the ultrapsherical polynomials shifted in $[x_0, x_1]$.

Theorem 3.2

If $W_n(x)$ are symmetric, the class of methods is P-stable for every order n.

Proof:

In this case, from (3.21) we have:

$$T_n(x_0) = (-1)^n \, T_n(x_1)$$

$$T'_n(x_0) = (-1)^{n-1} \, T'_n(x_1)$$

Then from (3.9), (3.22), (3.23)

$$\mu_1 = (-1)^n \left(\lambda^2 T_n^2(x_0) - T'^2_n(x_0)\right) / \left(\lambda^2 T_n^2(x_0) + T'^2_n(x_0)\right)$$

$$\mu_2 = (-1)^n \left(2\lambda \, T_n(x_0) \, T'_n(x_0)\right) / \left(\lambda^2 T_n^2(x_0) + T'^2_2(x_0)\right)$$

Therefore

$$|\mu|^2 = \mu_1^2 + \mu_2^2 = 1$$

and the thesis is proved.

REFERENCES

[1] M.R. Crisci, E. Russo - "A-stability of a class of methods for the numerical integration of certain linear systems of ordinary differential equations" - Math. Comp. vol. 38 (1982) pp. 431-435.

[2] M.R. Crisci, E. Russo - "An extension of Ortiz' recursive formulation of the Tau method for certain linear systems of ordinary differential equations" - Math. Comp. vol. 39 (1983) pp. 27-42.

[3] M.R. Crisci, E. Russo, A. Vecchio - "Periodicity intervals of the tau method for the numerical integration of certain second order differential equations" - Rendiconti della Accademia di Scienze Fisiche e Matematiche, Serie IV, vol. LIV (1987).

[4] J.D. Lambert - "Stiffness" in Computational Techniques for Ordinary Differential Equations - J. Gladwell, D.K. Sayers Ed. Academic Press, New York (1980) pp. 16-46

[5] C. Lanczos - "Trigonometric interpolation of empirical and analytical functions" - J. Math. Phys. vol.17 (1938) pp. 123-199

[6] C. Lanczos - "Applied Analysis" - Prentice Hall, New Jersey (1956).

[7] C. Lanczos - "Legendre versus Chebyshev polynomials" in Topics in Numerical Analysis - J.J.H. Miller Ed. Academic Press, New York, (1973) pp. 191-201.

[8] E.L. Ortiz - "The tau method" - SIAM J. Numerical Analysis, vol. 6 (1969), pp. 480-492

[9] E.L. Ortiz - "Step by step Tau method" - Comp. & Math. with Appl. vol. 1 (1975) pp. 381-392.

[10] K. Wright - "Some relationships between implicit Runge-Kutta collocation and Lanczos Tau method and their stability properties" - BIT vol. 10 (1970) pp. 217-227.

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HIGHER INTEGRABILITY OF THE GRADIENT OF MINIMA OF CERTAIN FUNCTIONALS

Nota di Nicola Fusco e Carlo Sbordone

Presentata dal Socio Carlo Sbordone

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Riassunto. Si dà un risultato di maggiore sommabilità per il gradiente di un minimo locale di un funzionale

$$\int_\Omega F(|Du|)\,dx$$

con $c_1 t^p - c_2 \le F(t) \le c_3 (1+t^q)$, $(p \le q)$.

Abstract. A higher integrability result is stated for the gradient of a local minimum of a functional

$$\int_\Omega F(|Du|)\,dx$$

with $c_1 t^p - c_2 \le F(t) \le c_3 (1+t^q)$, $(p \le q)$.

Let us consider the functional

$$I(\Omega, v) = \int_\Omega F(|Dv|)\,dx$$

where Ω is a bounded open set in R^n, $v: \Omega \dashrightarrow R^N$, $(n,N\geq 1)$ and the integrand $F:[0,+\infty) \dashrightarrow [0,+\infty)$ is a convex function.

The minima of these functionals have been extensively studied when F verifies control assumptions of the type

$$(1) \qquad c_1\, t^p - c_2 \leq F(t) \leq c_3\, (1+t^q), \qquad \forall\, t>0$$

with p=q. ([3],[4],[6]).

In this note we state a result concerning higher integrability of the gradient Du of a minimizer of I in the case q>p.

We say that $u \in W^{1,1}_{loc}(\Omega, R^N)$ is a minimizer of I(v) iff, $\forall\ \psi \in W^{1,1}(\Omega, R^N)$
with $\operatorname{supp} \psi \subset\subset \Omega$,

$$I(\operatorname{supp} \psi, u) \leq I(\operatorname{supp} \psi, u+\psi).$$

We have the following

THEOREM Let $F:[0,+\infty) \longrightarrow [0,+\infty)$ be an increasing convex function with $F(0)=0$, such that

$$(2) \qquad p\, F(t) \leq tF'(t) \leq q\, F(t) \qquad \text{for a.e. } t>0$$

where $nq/(n+q) < p \leq q$. Then, if $u \in W^{1,1}_{loc}(\Omega, R^N)$ is a minimizer of I(v), there exists r>1 such that $F(|Du|) \in L^r_{loc}(\Omega)$

An example of a function verifying the assumptions of the theorem for any n is :

$$F(t) = t^p \log(1+t) \qquad (p>1).$$

REMARK Condition (2) is equivalent to suppose $F(t)/t^p$ increasing and $F(t)/t^q$ decreasing, and it implies that F(t) satisfies the growth conditions (1), but it may happen that the exponents p,q appearing in (2) are not necessarily the best ones in order (1) to hold.

For example, the convex function

$$F(t) = \begin{cases} e\,t^3 & \text{if } 0<t<e \\ t^{4+\operatorname{sen}\log\log t} & \text{if } t>e \end{cases}$$

verifies (2) with $p=4-\sqrt{2}$ and $q=4+\sqrt{2}$ and verifies (1) with p=3 and q=5.
Moreover, if $4-\sqrt{2} < r < 4+\sqrt{2}$, $F(t)/t^r$ is neither definitely increasing nor decreasing.

REMARK 2 If p>n, then, in the theorem, q can be any number such that $q \geq p$ and the result also implies local Holder continuity for the minimizers.

The proof of the theorem is obtained combining a suitable version of Caccioppoli inequality ([6]) with the following extension of the so called Gehring lemma.

LEMMA Let $A: [0,+\infty) \to [0,+\infty)$ be an increasing convex function such that, for $1<p\leq q$

$$p\,A(t) \leq t\,A'(t) \leq q\,A(t) \qquad \text{for a.e. } t>0.$$

If $f \in L^1_{loc}(\Omega)$ is a non negative function such that, for any ball $B_\rho \subset \Omega$

$$⨍_{B_{\rho/2}} A(f)\,dx \leq c_1 A\Big(⨍_{B_\rho} f\,dx \Big) + c_2,$$

then there exists r>1 such that

$$⨍_{B_{\rho/2}} A^r(f)\,dx < c'_1\, A^r\left(⨍_{B_\rho} f\,dx \right) + c'_2$$

where the constants c'_i do not depend on f.

Let us note that the proof of the lemma can be obtained by an argument which is different from those of [1] ,[2] , [4], [5] .

REFERENCES

[1] B.BOJARSKI-T.IWANIEC, "Analitical foundations of the theory of quasi-conformal mappings in R^n " Annales Ac. Sci. Fen. 8 (1983) , p. 257-324.

[2] N.FUSCO-C.SBORDONE, " Higher integrability from reverse Jensen inequalities with different supports", vol. dedicated to E. De Giorgi, Birkauser (to appear).

[3] M.GIAQUINTA-E.GIUSTI, " On the regularity of the minima of variational integrals", Acta Math. 148, (1982), p.31--46.

[4] M.GIAQUINTA-G.MODICA, "Regularity results for some classes of higher order non linear elliptic systems", J.Reine Angew. Math. 311/312, (1979), p. 145-169.

[5] T.IWANIEC," Gehring's reverse maximal function inequality", in Approximation and Function Spaces, Gdansk (1979), North Holland, (1981).

[6] C.SBORDONE, "On some integral inequalities and their applications to the Calculus of Variations", Boll. UMI, Analisi Funzionale e Applicazioni, Serie VI, vol. V, C, 1, (1986), p. 73-94.

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ON SOME INTEGRAL INEQUALITIES

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Presentata dal Socio Carlo Sbordone
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Riassunto

Utilizzando un lemma di Paley-Zygmund, si prova, sotto opportune ipotesi, che un peso w che soddisfa la condizione A_φ di Kerman e Torchinsky, soddisfa una condizione A_{φ_δ}. Con la stessa tecnica si riottiene un lemma di Gehring sulle disuguaglianze di Hölder al contrario.

Abstract

By using a Paley-Zygmund's lemma, we prove, under suitable assumptions, that a weight w, satisfying the A_φ condition of Kerman and Torchinsky, satisfies a A_{φ_δ} condition. We reobtain a Gehring's lemma on reverse Hölder inequalities.

§1. INTRODUCTION

In [4] Kerman-Torchinsky consider an inequality of the type $(k>1)$

$$(1.1) \qquad \left(\fint_Q w\,dx\right) \varphi\left(\fint_Q \varphi^{-1}(1/w)\,dx\right) \le k$$

for any cube $Q \subset R^n$ with sides parallel to the axes, where φ is an increasing, non negative function on $[0,\infty]$, with $\varphi(0)=0$ and $w: R^n \to [0,\infty)$ is a weight, where we set $\fint_Q g = (1/|Q|)\int_Q g$.

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They prove that, under suitable assumptions on φ ,(1.1) is a necessary and sufficient condition in order that the inequality

(1.2) $\int_{R^n} \Phi(Mf)\ wdx \le C_\Phi \int_{R^n} \Phi(|f|)\ wdx$

holds for any measurable f, where

(1.3) $\Phi(t) = \int_0^t \varphi(s)ds.$

In this way they extend a well known theorem by Muckenhoupt [6] relative to the case $\varphi(t) = t^{p-1}$, $p>1$.

Namely, they assume that the convex function $\Phi(t)$ verifies the Δ_2-condition together with its conjugate (see the definitions in section 2), and they prove that (1.1) and (1.2) are equivalent and imply the condition:

(1.4) $\exists\ \eta>0 :\ (⨍_Q wdx)\ \varphi_\eta\ (⨍_Q \varphi_\eta^{-1}\ (1/w)\ dx) \le k'$

for any parallel cube $Q \subset R^n$, where

$$\varphi_\eta(s) = \varphi\ (s^{1/(1+\eta)}).$$

The aim of this paper is to consider also functions Φ whose conjugate does not verify the Δ_2-condition. It is well known that, in this more general situation, the implication (1.1)$\Rightarrow$(1.2) does not hold (see e.g.[7]); nevertheless, we prove that the implication (1.1)$\Rightarrow$(1.4) continues to hold,under rather general assumptions (see section 3).

An example of a function which verifies our assumptions is:

$\Phi(t) = t^p(1+a\ \log_+(1/t))$, $p>1, a<\min\ \{1,p(p-1)/(2p-1)\}$.

In section 4 we consider also some cases in which from (1.1) it is possible to deduce that the weight w satisfied the A_p-condition, for a p>1; for example $\Phi(t)=t\log(1+t)$.

We get these results using a suitable extension of a classical Paley-Zygmund lemma.

In section 5 we prove that, following the same lines of the proof, we reobtain also a well known Gehring's lemma about reverse Hölder inequalities.

§2.

In this section we state some lemmas which are similar to a classical Paley-Zygmund lemma (see [8] and also [2] for some generalizations).

Lemma 2.1 [2]

Let g be a measurable function on a measure space $(X,\mathcal{B},\mu)$ and $E\subset X$ a measurable set with $\mu(E)>0$ *and* $g\geq 0$ *on E. Further let A, B>0 verify:*

(i) $(⨍_E g\, d\mu) \geq A>0$;

(ii) $(⨍_E g^p\, d\mu)^{1/p} \leq B$, where p>1.

Then, for any $\delta \in (0,1)$:

(2.1) $\mu\{x\in E : g(x) > \delta A\} \geq \mu(E)\ [(1-\delta)A/B]^{p/(p-1)}$.

Lemma 2.2. Let φ $[0,+\infty)\rightarrow[0,+\infty)$ *be an increasing function and let* $v:E\rightarrow[0,+\infty)$ *be a measurable function on* $(X,\mathcal{B},\mu)$, $E\in\mathcal{B}$.

Let $A,B>0$ verify:

(i) $\varphi(⨍_E v d\mu) \geq A$;

(ii) $⨍_E (1/\varphi(v))\, d\mu \leq B$.

Then, there exists $\delta_o<1$ such that $\forall\delta\varepsilon(0,\delta_o)$:

(2.2) $\mu\{x\varepsilon E: v(x) > \delta\varphi^{-1}(A)\} \geq \mu(E)[1-B\varphi(\delta\varphi^{-1}(A))]$.

Proof. For $\delta>0$ set $E_\delta = \{x\varepsilon E: v(x)>\delta\, \varphi^{-1}(A)\}$.

Then, for $x\varepsilon E-E_\delta$ we have $\varphi(v(x)) \leq \varphi(\delta\varphi^{-1}(A))$

and so:

(2.3) $\int_{E-E\delta} (1/\varphi(v))\, d\mu \geq \mu(E-E_\delta)/[\varphi(\delta\varphi^{-1}(A))]$.

From (2.3) and (ii) it follows:

$$\mu(E-E_\delta)/[\mu(E)\ \varphi(\delta\varphi^{-1}(A))] \leq ⨍_E (1/\varphi(v))\, d\mu \leq B$$

and so:

$$\mu(E-E_\delta) \leq \mu(E)\ B\varphi(\delta\varphi^{-1}(A)).$$

The result follows choosing $\delta_o = \min\{1, \varphi^{-1}(1/B)/\varphi^{-1}(A)\}$.

Lemma 2.3. Let $\varphi:[0,+\infty)\to[0,+\infty)$ be an increasing function and let $w: E\to[0,+\infty)$ be a measurable function on $(X,\mathcal{B},\mu)$, $E\varepsilon\mathcal{B}$. Further let $A, B>0$ verify:

(i) $⨍_E w d\mu \geq A>0$;

(ii) $\varphi(⨍_E \varphi^{-1}(1/w)\, d\mu) \leq B$.

Then there exists $\delta_o< 1$ such that, for any $\delta<\delta_o$:

(2.4) $\mu\{x\ \varepsilon E: w(x)>\delta A\} \geq \mu(E)[1-[\varphi^{-1}(B)/\varphi^{-1}(1/\delta A)]]$.

Proof: For $\delta>0$ set $E_\delta = \{x\ \varepsilon\ E: w(x) > \delta A\}$

Then, for $x\ \varepsilon E-E_\delta$ we have $\varphi^{-1}(1/w) \geq \varphi^{-1}(1/\delta A)$

and so:

(2.5) $\qquad \int_{E-E_\delta} \varphi^{-1}(1/w)\,d\mu \geq \mu(E-E_\delta)\varphi^{-1}(1/\delta A).$

From (2.5) and (ii) it follows:

$$[\mu(E-E_\delta)/\mu(E)]\ [\varphi^{-1}(1/\delta A)] \leq ⨍_E \varphi^{-1}(1/w)\,d\mu \leq \varphi^{-1}(B)$$

and so:

$$\mu(E-E_\delta) \leq \mu(E)[\ \varphi^{-1}(B)/\varphi^{-1}(1/\delta A)].$$

The result follows choosing $\delta_o = \min\{1, 1/AB\}$.

<u>Remark 2.1</u>. If $\varphi(t)=t^{p-1}$ (i) and (ii) become:

$$A \leq ⨍_E w\,d\mu \quad ; \qquad B \geq [⨍_E w^{-1/(p-1)}\,d\mu]^{p-1}$$

and (2.4): $\mu\{x\in E : w(x)>\delta A\} \geq \mu(E)\ [1-(\delta\ AB)^{1/(p-1)}] \qquad \forall\delta<1/AB.$

We say that the convex function

$$\Phi(t) = \int_o^t \varphi(s)\,ds$$

where $\varphi(s)$ is an increasing function defined on $[0,+\infty)$; with $\varphi(0+)=0$ verifies the Δ_2-condition if there exists $B>1$ such that:

$$\Phi(2t) \leq B\Phi(t) \quad \forall t>0.$$

The conjugate of $\Phi(t)$ is the convex function $\tilde{\Phi}(t)$ defined by

$$\tilde{\Phi}(t) = \int_o^t \varphi^{-1}(s)\ ds.$$

The following lemma by Coifman-Fefferman will be crucial for the sequel [1].

<u>Lemma 2.4. Let</u> $w\in L^1_{loc}(R^n,dx)$ <u>and suppose that the condition</u>

(A'_∞) $\quad |\{x\in Q : w(x)>\beta m_Q(w)\}|>\alpha|Q| \qquad$ where $m_Q(w) = ⨍_Q w\,dx$

holds for some positive constants α, β, for any cube Q with sides parallel to the axes. Then there exists $c, \eta > 0$ independent of Q such that the reverse Hölder inequality

$$\left(⨍_Q w(x)^{1+\eta}\, dx\right)^{1/(1+\eta)} \leq c\left(⨍_Q w(x)\, dx\right)$$

holds for all cubes Q.

§3.

Theorem 3.1. Let φ be an increasing function, verifying the Δ_2-condition, with $\varphi(0)=0$ such that:

(3.1) $\varphi(st) \leq c\, \varphi(s)\, \varphi(t) \qquad \forall s \in (0,1), \forall t>0$

and let w be a weight verifying:

(3.2) $\left(⨍_Q w dx\right) \varphi\left(⨍_Q \varphi^{-1}(1/w)\, dx\right) \leq k$

for any cube Q with sides parallel to the axes, with $k \geq 1$. Then there exists $\eta>0$ such that, setting $\varphi_\eta(t) = \varphi(t^{1/(1+\eta)})$, (3.2) holds with φ replaced by φ_η

Proof. Let $v = \varphi^{-1}(1/w)$ then (3.2) becomes:

$$\left(⨍_Q (1/\varphi(v))\, dx\right) \quad \varphi\left(⨍_Q v dx\right) \leq k \quad \forall \text{ cube } Q.$$

Using lemma 2.2 with $A = \varphi\left(⨍_Q v dx\right)$, $B = ⨍_Q (1/\varphi(v))\, dx$, we have $AB \leq K$ and

(3.3) $|\{x \in Q: v(x) > \delta\varphi^{-1}(A)\}| \geq |Q| \; [1 - B\varphi(\delta\varphi^{-1}(A))]$

for $\delta < \min\{1, \varphi^{-1}(1/ck)\}$.

In fact (3.3) holds for δ such that $\varphi(\delta\varphi^{-1}(A)) < 1/B$; now, if $\delta < \varphi^{-1}(1/ck)$, we have:

(3.4) $\varphi(\delta) < 1/ck \leq 1/cAB$

and from (3.1):

(3.5) $\varphi(\delta\varphi^{-1}(A)) \le c\ \varphi(\delta) A$

that is, $\varphi(\delta) > (1/cA)\ \varphi(\delta\varphi^{-1}(A))$.

Therefore, putting together (3.4) and (3.5) we get:

$$\varphi(\delta\varphi^{-1}(A)) < 1/B.$$

Using (3.5) we have:

$$|\{x\in Q:\ v(x) > \delta \fint_Q v dx\}| \ge |Q|\ [1-cBA\ \varphi(\delta)] \ge |Q|[1-c\varphi(\delta)k]$$
$$\text{for } \delta<\min\{1,\varphi^{-1}(1/ck)\}$$

that is a condition (A'_∞).

Using lemma 2.4 we have:

$$\fint_Q v^{1+\eta} \le c(\fint_Q v)^{1+\eta}$$

for all cube Q with sides parallel to the axes;

then, substituting $v=\varphi^{-1}(1/w)$ and $\varphi_\eta(t)=\varphi(t^{1/(1+\eta)})$:

$$(\fint_Q \varphi_\eta^{-1}(1/w)dx)^{1/(1+\eta)} \le \bar{c}\ (\fint_Q \varphi^{-1}(1/w)dx)$$

and also using our assumption (3.2)

(3.6) $$\varphi_\eta(\fint_Q\varphi_\eta^{-1}(1/w)dx) \le \varphi(\bar{c} \fint_Q\varphi^{-1}(1/w)dx) \le c'\varphi(\fint_Q\varphi^{-1}(1/w)dx) \le$$
$$\le c'k/\fint_Q w\ .$$

In conclusion, we have:

$$(\fint_Q wdx)\ \varphi_\eta(\fint_Q\varphi_\eta^{-1}(1/w)dx) \le c'k.$$

<u>Example.</u> The function $\Phi(t) = t^p(1+a\log_+(1/t)), p>1$ is submultiplicative, increasing and convex when $a< \min\{1,p(p-1)/(2p-1)\}$.

§4. Now we consider another assumption on φ

$\varphi(cst) \geq \varphi(s)\ \varphi(t) \qquad \forall s\varepsilon(0,1),\ t>0$, that is:

$c\varphi^{-1}(u)\ \varphi^{-1}(v) \geq \varphi^{-1}(uv) \qquad \forall u\varepsilon(0,\varphi(1)),\ v>0$

and we prove that the A_φ condition implies an A_p condition for a $p>1$.

Theorem 4.1. Let φ be an increasing function, $\varphi(0)=0$ such that

(4.1) $\quad \varphi(cst) \geq \varphi(s)\ \varphi(t) \qquad \forall s\varepsilon(0,1),\ t>0$

and let w be a weight verifying:

(4.2) $\quad (⨍_Q w\,dx)\ \varphi(⨍_Q \varphi^{-1}(1/w)\,dx) \leq k.$

for all cube Q with sides parallel to the axes, with $k\geq 1$.

Then, there exists $p>1$ such that $w\varepsilon A_p$.

Proof. Using lemma 2.3 with $A = ⨍_Q w\,dx$ and $B=\varphi(⨍_Q\varphi^{-1}(1/w)\,dx)$, $E=Q$, we have $AB\leq k$ and

(4.3) $\quad |\{x\varepsilon Q: w(x)>\delta A\}| \geq |Q|\ [1-[\varphi^{-1}(B)/\varphi^{-1}(1/\delta A)]]$

for $\delta\leq\varphi(1/c)/k$.

In fact (4.3) holds for $\delta<1/AB$;

now if $\delta<\varphi(1/c)/k$, we have $\varphi^{-1}(\delta)<\varphi^{-1}(\varphi(1/c)/k)$

and, from (4.1), $\varphi^{-1}(\delta)<\varphi^{-1}(1/k)\leq \varphi^{-1}(1/AB)$

that is, $\delta< 1/AB$.

From (4.1) we have:

(4.4) $[\varphi^{-1}(B)]/[\varphi^{-1}(1/\delta A)]=[\varphi^{-1}(\delta AB/\delta A)]/[\varphi^{-1}(1/\delta A)]\leq c\varphi^{-1}(\delta AB)$

Substituting (4.4) in (4.3):

$(A'_\infty)\quad |\{x\varepsilon Q: w(x)>\delta m_Q(w)\}| \geq [1-c\varphi^{-1}(\delta AB)]\ |Q| \geq [1-c\varphi^{-1}(\delta k)]\,|Q|$

for $\delta< \varphi(1/c)/k$.

From lemma (2.4), there exists an $\eta>0$ such that:

$$⨍_Q w^{1+\eta}\, dx \le c\,(⨍_Q w)^{1+\eta}$$

for all cube Q with sides parallel to the axes.

<u>Example</u>: The function $\Phi(t)=t^p \log(1+t)$, $p\ge1$ is supramultiplicative and for $p=1$ its conjugate Φ doesn't verify the Δ_2-condition.

§5. In this section we prove that, following the same lines of the proofs of §3 and §4, that is via Paley-Zygmund's Lemma, (Lemma 2.1), we reobtain a classical Gehring's theorem about reverse integral inequalities. For $1<p<+\infty$ a nonnegative weight $g \in L^p_{loc}(R^n)$ verifies the G_p condition [3] if there exists a $k>1$ such that

$$(⨍_Q g^p\, dx)^{1/p} \le k\,(⨍_Q g\,dx) \tag{5.1}$$

for each cube Q with sides parallel to the axes. We give another proof of the following theorem [3]:

<u>Theorem 5.1. Let</u> g <u>be a weight verifying the</u> G_p <u>condition, then</u> g <u>has higher integrability; that is, there exists a</u> $\eta>0$ <u>such that:</u>

$$⨍_Q g^{p(1+\eta)}\, dx \le c\,(⨍_Q g^p)^{1+\eta}. \tag{5.2}$$

<u>Proof</u>. We observe that, from Hölder inequality and from (5.1) we have:

$$⨍_Q g\,dx \le (⨍_Q g^p\, dx)^{1/p} \le k\,(⨍_Q g\,dx) \tag{5.3}$$

for each cube with sides parallel to the axes.

Using lemma 2.1 with

$$A = ⨍_Q g\,dx, \qquad B = k ⨍_Q g\,dx, \quad E=Q$$

we get, from (5.3), B=kA and

(5.4) $|\{x\in E : g(x) > \delta A\}| \geq |Q|\,[(1-\delta)A/B]^{p/(p-1)} = |Q|\,[(1-\delta)/k]^{p/(p-1)}$.

Therefore, from (5.1), we have:

$$(m_Q(g))^p = (⨍_Q g)^p \geq (1/k^p)\ (⨍_Q g^p dx) = (1/k^p)\ m_Q\ (g^p)$$

and then:

(5.5) $|\{x\in Q : g^p\ (x) > \delta^p [m_Q(g)]^p\}| \leq |\{x\in Q : g^p(x)\ > (\delta/k)^p [m_Q(g^p)]\}|$
for any $\delta\in(0,1)$.

From (5.4) and (5.5) we get a A'_∞ condition for g^p:

(A'_∞) $|\{x\in Q : g^p(x)\ > (\delta/k)^p [m_Q(g^p)]\}| \geq |Q|\,[(1-\delta)/k]^{p/(p-1)}$.

Using lemma (2.4) we get the desired conclusion.

References:

[1] R. Coifman, C. Fefferman - *Weighted norm inequalities for maximal functions and singular integrals.* Studia Math. 51 (1974) 241-250

[2] L. D'Apuzzo - (*to appear*)

[3] F. W. Gehring - *The L^p integrability for the partial derivatives of a quasiconformal mapping.* Acta Math. 130(1973 265-277

[4] R.A. Kerman, A. Torchinsky - *Integral inequalities with weights for the Hardy maximal function.* Studia Math. 71, (1982) 277-284

[5] A. Kufner, O.John, S. Fucik - *Functions Spaces.* Noordhoff Leyden, 1977

[6] B. Muckenhoupt - *Weighted norm inequalities for the Hardy maximal function.* T.A.M.S. 165 (1972) 207-226

[7] G. Palmieri - *Un approccio alla teoria degli spazi di traccia relativi agli spazi di Orlicz-Sobolev.* B.U.M.I.(1979).

[8] A. Zygmund - *Trigonometric Series.* Cambridge University Press, N.Y. (1959)

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LATE RATHER THAN EARLY TERTIARY DEFORMATION
OF EXTERNAL DINARIDES.
STRATIGRAPHIC EVIDENCE FROM MONTENEGRO

Nota di Rajka RADOIČIĆ, Paola DE CAPOA & Bruno D'ARGENIO
Presentata dal Socio ordinario Bruno D'Argenio.

Adunanza del 6 maggio 1989

Riassunto. Vengono presentati i primi dati biostratigrafici che ringiovaniscono dal Paleogene al Miocene medio - (?) superiore l'età dei depositi terrigeni coinvolti nei sovrascorrimenti delle Dinaridi Esterne. Da ciò deriva che le deformazioni che hanno colpito questo settore dell'Oro= gene Periadriatico sono sostanzialmente coeve di quelle delle zone esterne dell'Appennino e del Sud Alpino.

Abstract. Biostratigraphic analysis of terrigenous sequences involved in thrust tectonics in the External Dinarides (Montenegro) reveals that these deposits have middle - (?) late Miocenic, rather than Paleogenic age. It follows that the orogenic deformation wich affected this sector of the Periadriatic Mountain Belt developed basically at the same time of that acting on the external zones of Apennines and Southern Alps.

1. Introduction

In the European approach to the study of mountain belts the inception of terrigenous deposits ("flysch") in the "geosynclinal" domains has been viewed, since the thirties, as forerunner of deformation, being the extrabasinal clastic influx considered as good evidence of dismantling of close relief, rised by compressional tectonics. Likewise in a nappe within a nappe pile, the youngest age proved at the top of a conformable "flysch" sequence suggests the timing of the overlying nappe emplacement.

In the last decades, together with a plate tectonic based geodynamic re-interpretation of the "alpine" belts of the Mediterranean, a vast literature has been produced on the structural organization of this region, while the relative timing of evolution has been continuously uptodated, according to the advancement of the modern biostratigraphy.

For the External Dinarides, western sector of the wide belt running from the Periadriatic Lineament to the Scutari-Pec line (KÖBER, 1929; AUBOUIN, 1963), an Early Tertiary age of deformation has been traditionally accepted (see, f.i., AUBOUIN, 1960, 1963; CELET, 1977; DERCOURT et al., 1986). Infact datations based mostly on foraminifera pointed out a late Eocene-Oligocene age for the topmost conformable terrigenous ("flysch") levels of known sequences, downwards grading into older Paleogene and Mesozoic layers, either of basinal or of shallow water facies.

In contrast, for the facing external sectors of the Apennines, as well as for more northern segments of the Periadriatic Belt (Northern Apennines and Southern Alps), Neogene timing of deformation has been since the late fifties largely recognized and documented, using the same tectono-sedimentary approach (e.g. ACCADEMIA DEI LINCEI, 1973; OGNIBEN et al., 1975).

In the present paper a number of biostratigraphic evidences, primarily based on coccolithes, are offered to show the Miocenic (rather than Paleogenic) age of terrigenous deposits sampled in Southern Montenegro, where they form part of the sequences of the External Dinarides nappe pile. We are aware of the influence that these findings (briefly and informally anticipated in RADOIČIĆ & D'ARGENIO, 1988, page 57; DE CAPOA et al., 1989) may have on a re-evaluation of the local as well regional tectonic history. As a matter of fact, it appears that a Miocene time interval of deformation for the whole external domains of Dinarides, Apennines and Southern Alps, calls for a new and better documented discussion on the tectonic evolution and mechanisms of the entire Periadriatic Region.

2. Geology of the Ulcinj District

The area discussed here after, already described in a few papers (LUKOVIĆ & PETKOVIĆ, 1952; VIDOVIĆ, 1965; BURIĆ, 1966; ROKSANDIĆ & ČANOVIĆ, 1971), is part of a 15 by about 50 Km NW-SE trending mountain group, rising up to some

1000 m above sea level between the Adriatic Sea and the Lake of Scutari (Skadarsko Jezero), W of the River Bojana (fig.1).

In the coastal sector of the above mountains, extending from the town of Bar to few kilometers SE of Ulcinj, the following four tectonic elements, trending NW-SE, may be singled out: Ulcinj-Mendre, Bijela Gora-Mavrijan, Možura-Brivska Gora and Volujica-Šaško Brdo (LUKOVIĆ & PETKOVIĆ, 1952; RADOIČIĆ & D'ARGENIO, 1988).

These structures consist of narrow overturned anticlines thrust to the SW, and are formed by Cretaceous-Eocene shallow water limestones, passing to Eocene-Oligocene pelagic limestones and marls and than to terrigenous deposits; the latter, formerly considered of Paleogenic age, are here dated as Miocene.

The terrigenous younger part of these sequences is mostly preserved in narrow belts, at lower elevation, where it may be seen to abut against or to steep below these structures, with complex tectonic relationships.

The paleogeographic domain from which the Ulcinj structures derive is the Adriatic Carbonatic Platform (D'ARGENIO et al., 1971), a Mesozoic-Early Tertiary realm of shallow water sedimentation. The predeformation history of this sector follows a pathway typical for the Periadriatic Region: (1) it individuated in the Late Triassic, (2) subsided sluggishly up to the Late Cretaceous (accumulating probably more than 3 Km of evaporitic and carbonatic deposits) and underwent short emersion (documented by bauxites and paleokarst) at the end of Mesozoic times; then (3) shallow water sedimentation again developed in the Early Tertiary for a short interval, followed by (4) drowning and transition to pelagic deposits which, in turn, underwent gradually but rapidly (5) terrigenous influx.

Our data here after mostly deal with the last two items (for other information the reader may refer to the above quoted papers and to RADOIČIĆ & D'ARGENIO, 1988).

3. Stratigraphic outline of Možura-Brivska Gora structure

The Mesozoic sequence of Ulcinj District is outcropping only from the Upper Cretaceous. Older strata are only known in the subsurface (ČANOVIĆ, 1965).

The Možura-Brivska Gora element, which is facing the Adriatic coast and is the most easily accessible (particularly along the Ulcinj-Bar roadcuts), provides a good idea of the stratigraphy of the whole area (fig. 2).

The lowest strata consist of no more than 60 m of:
- (a) rudistid limestones and dolomitized limestones, including stromatolitic levels and breccias, indicating a

very shallow water environment of deposition. Rudistid (Bournonia, Durania, Katzeria, Fundinia, Gorjanovicia, Biradiolites) and foraminifera (Accordiella, Keramosphaeri=na, Scandonea, Minouxia, Dicyclina) suggest a Turonian (?) - Coniacian age.

- (b) The last few meters of this interval are almost bar=ren, while Microcodium is often developing, together with some evidence of karstic solution. Lenses and pockets of bauxite then follow to mark a disconformity which may be observed at regional scale.

The next marine deposits are already of Tertiary age. They are formed by about 50 m of well bedded limestones which upwards include two intervals:

- (c) fine grained limestones (mudstone to wackestone with dasycladaceans and foraminifera, among which rare Spirolina, then packstone with Nummulites sp. and Fabiania casis) passing to foraminiferal limestones (packstone with Disco=cyclinids and rare, small Nummulitids);
- (d) nodular limestones, with pelagic forams and small bioclasts (Discocyclinids and Briozoans) which are topped by a distinct hardground. Above follows:
- (e) marly limestones (about 20 m) grading upwards into thick (about 150 m) alternations of sandstones and siltsto=nes capped by about 1 m of calcarenites and fine grained calcirudites, which here form the topmost stiff layer of the sequence, exhumed by weathering.

The intervals from (c) to (e) exhibit a clear trend from a shallow regime (more restricted to some how open shelf type) into a pelagic environment of deposition, through a drowning episode and a compositional change in the biogenic component.

In a few meters, and with apparent continuity, terrigenous admixture to the clayey deposits increases, giving rise to the interval (e), more than 150 m thick (at least 500 m according to LUKOVIĆ & PETKOVIĆ, 1952) and formed by alternations of silty clays and sandstones and capped by calcarenites.

According to the previous Authors (PAPP & AMSEL, 1961; LUKOVIĆ & PETKOVIĆ, 1952), who have also studied samples from the deep borehole "Ulcinj" (ČANOVIĆ & DŽODŽO, 1958), this part of the sequence is of pre-Neogene age, or even of pre-Oligocene age (PAVIĆ, 1970 and Geological Map of Jugoslavia, 1:500.000 scale, 1970).

4. Biostratigraphy

We have made a preliminary biostratigraphic analysis of the calcareous nannofossils of the Možura se= quence, with particular regard to the last interval.

From a first collecting we have examined 3 samples from the lower part of the interval (e), well spaced among them, and 4 samples located in the upper part of the same interval. The following is a brief account about their study (see Tables I to IV).

A first sample (n. 025.801) was collected from the base of the interval (e).
Markers: Cyclicargolithus floridanus (ROTH & HAY) BUKRY, Isthmolithus recurvus DEFLANDRE, Pedinocyclus larvalis (BUKRY & BRAMLETTE) LOEBLICH & TAPPAN, Reticulofenestra oamaruensis (DEFLANDRE & FERT) STRADNER, Sphenolithus pseudoradians BRAMLETTE & WILCOXON.
Age: Upper Eocene.
Biozone: upper part of CP 15b, Isthmolithus recurvus subzone OKADA & BUKRY 1980 = NP 20, Sphenolithus pseudoradians zone MARTINI 1971 = P 16p.p. BLOW 1969, BERGGREN et al. 1985.

A second sample (n. 025.802) was collected few me= tres above, in the lower part of the same interval.
Markers: Helicosphaera euphratis HAQ, Pyrocyclus hermosus ROTH & HAY, Reticulofenestra hillae BUKRY & PERCIVAL, Ericsonia obruta PERCH-NIELSEN (acme zone).
Age: Lower Oligocene.
Biozone: CP 16a - CP 16b, limit between Coccolithus subdi= stichus e C. formosus subzones OKADA & BUKRY 1980 = NP 21, Ericsonia subdisticha zone MARTINI 1971 = upper part of P 18 BLOW 1969, BERGGREN et al. 1985.

A third sample (n. 025.803) comes from the middle part of the interval (e).
Markers: Coccolithus miopelagicus BUKRY, Cyclicargolithus abisectus (MÜLLER) WISE, Discoaster adamanteus BRAMLETTE & WILCOXON, Discoaster calculosus BUKRY, Helicosphaera recta HAQ, Helicosphaera truempyi BIOLZI & PERCH-NIELSEN.
Age: Upper Oligocene.
Biozone: CP 19b, Dictyococcites bisectus subzone OKADA & BUKRY 1980 = NP 25, Sphenolithus ciperoensis zone MARTINI 1971 = P 22/N 3 BLOW 1969, BERGGREN et al. 1985.

Four samples (from 025.804 to 807) were collected in the upper part of the interval (e), aiming at sampling the most recent part of the terrigenous sequence. All 4 sam= ples have yelded Miocenic nannofloras as follows:
Markers: Discoaster bollii MARTINI & BRAMLETTE, Discoaster calcaris GARTNER, Sphenolithus abies DEFLANDRE & FERT, Sphenolithus grandis HAQ & BERGGREN.
Age: Upper Serravallian.
Biozone: upper part of CN 6, Catinaster coalitus zone OKADA

& BUKRY 1980 = upper part of NN 8, Catinaster coalitus zone MARTINI 1971 = N 13 BLOW 1969, BERGGREN et al. 1985.

It may also be added to the above data that one from the 4 samples (n. 025.807) includes also Discoaster cf. mendomobensis WISE, whose first occurrence is in CN 8b, Discoaster neoerectus subzone OKADA & BUKRY 1980 = upper part of NN 10, Discoaster calcaris zone MARTINI 1971. Should the last species be found in the analytical work now in progress, also the Lower Tortonian would be documented in the studied sequence.

Study in progress in the Ulcinj district also in=cludes detailed sampling of the terrigenous part of the se=quence (interval e) in a section cropping out North of Gola Glava, between Možura and Šaško Brdo structures (fig. 1). The following preliminary data from this section may be anticipated.

The base of the interval (e) has yelded a Late Eo=cene age, while the Oligocene develops for more than 30-35 m. The base of the Miocene deposits (Helicosphaera gertae BUKRY) follows at about 1 m, and is already rich in fine terrigenous material. The next 97 m have yelded Miocenic nannofloras, spanning between Aquitanian and Upper Serraval=lian - (?Lower Tortonian).

Similar results also came from scattered samples collected around the Kotor Bay (road from Herceg Novi to Gruda) as well as from other sites along the Adriatic coast or even from the more internal High Karst (Dinaric Carbonate Platform) zone. We note also that further findings could disclose younger age too, because very high reworking precludes often a quick age determination.

Reworking deserves a final comment, because it increases upwards steadly in the lower part of the interval (e) and very rapidly in its terrigenous middle-upper part. The ratio of non-reworked to reworked taxa (considering only one specimen per each species) changes from 3:1 (Upper Eoce=ne - Lower Oligocene) to 1:1 (Upper Oligocene - Lower Mioce=ne) to 3:1 (Middle - ?Upper Miocene).

We also counted 500 nannofossils from a sample of Upper Serravallian age, where the non reworked/reworked ra=tio reaches 1:16 (28 non-rw. to 472 rw. specimens, excluding long ranging species, like Coccolithus pelagicus or C. mio=pelagicus).

The above figures suggest for nannofossils cumula=tive effects of reworking merely with elapsing time in the terrigenous basins of the migrating Dinaric foredeep, a process not yet well understood in its subtle mechanism.

5. Final remarks

The External Dinarides are traditionally considered as an Alpidic segment formed mostly during the Early Tertiary, consistently with the Paleogenic age assigned to the terrigenous deposits conformably topping their folded and thrust sequences as well as with the time of deformation of the adjacent Hellenides and Alps (AUBOUIN 1963, CELET 1977, DERCOURT et al. 1986).

A biostratigraphic study of these sequences has on the contrary revealed Miocenic nannofossils amidst higly reworked older fossils. Younging the age of the terrigenous deposits, changes the timing of compressional deformation in the region, thus offering different tectonic scenarios to the regional geodynamic restoration and models.

Therefore, while the Southern Apennines, thrust during the Neogene onto the Apulian Foreland, were formerly considered as facing a Dinaric segment already deformed du= ring the Paleogene with opposite vergence, we need now to regard also the Southern Dinaric fold and thrust belt deve= loping during the late Tertiary times. It seems reasonable to assume that similar situations may be found also more to the north.

The general implications of these new time rela= tions among the Periadriatic orogenic segments call for very high mobility and decoupling at upper lithospheric levels. The above new boundary conditions appear very promising in the present circumstances of impending opera= tions for deep seismic sounding in Southern Italy.

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REFERENCES

ACCADEMIA NAZIONALE DEI LINCEI, 1973 - Moderne vedute sulla geologia dell'Appennino. Quaderno n. 183, pp. 454. Roma.

AUBOUIN, J., 1960 - Essai sur l'ensemble Italo-Dinarique et ses rapports avec l'Arc Alpin. Bull. Soc. Géol. France, v. 2, p. 487-526.

AUBOUIN, J., 1963 - Esquisse paléogéographique et structu= rale des chaines alpines de la Méditérranée moyenne. Geol. Rundsch., v. 53, p. 480-534.

BERGGREN, W.A., KENT D.V. & VAN COUVERING, J.A., 1985 - Neogene geochronology and chronostratigraphy. In: SHELLING, N.J. (Ed.), The chronology of the geological record, British Geol. Surv. London, Mem. n. 10.

BLOW, W.H., 1969 - Late Middle Eocene to Recent planktonic foraminiferal Biostratigraphy. Proceed. I Int. Conf. Plankt. Microfossils, v. 1, Geneva 1967.

BURIĆ, P., 1966 - Geologija lezista boksita Crne Gore. Pos. Izd. Geol. glasnika, v. 8, Sarajevo.

ČANOVIĆ, M., 1965 - Biostratigrafske odlike kredne serije u profilu busotine Mozural-1 kod Ulcinja. Geol. Glasnik, v. 4, Titograd.

ČANOVIĆ, M. & DŽODŽO, R., 1958 - Prethodna beleska o mikro=fauni oligocena iz busotine Us-6 kod Ulcinja. Geol. Glasnik, v. 2, Titograd.

CELET, P., 1977 - The Dinaric and Aegean Arcs: the Geology of the Adriatic. In: NAIRN A.E.M. et al. (Eds.), The Eastern Mediterranean, p. 215-216. Plenum Press, N.Y.

D'ARGENIO, B., RADOIČIĆ, R. & SGROSSO, I., 1971 - A paleo=geographic section through the Italo-Dinaric External Zones during Jurassic and Cretaceous times. Nafta, v. 22/4-5, p. 195-207, Zagreb.

D'ARGENIO, B., HORVATH, F. & CHANNEL, J.E.T., 1980 - Paleo=tectonic evolution of Adria, the African Promontory. Mem. B.R.G.M. n. 115 (Coll. C5: Géologie des Chaines Alpines issues de la Téthys, 26° CGI), p. 331-351.

DE CAPOA, P., RADOIČIĆ, R. & D'ARGENIO, B., 1989 - Evidenze biostratigrafiche di tettogenesi miocenica nelle Dinaridi Esterne del Montenegro. Confer. Scient. 1989, Dipartimenti Geof. e Vulcanol., Paleontologia e Scienze della Terra, Università di Napoli, p. 93-95, Napoli.

DERCOURT, J., ZONENSHAIN, L.P., RICOU, L.E., KAZMIN, V.G. , LE PICHON, X. and 14 others, 1986 - Geological evolution of the Tethys Belt from the Atlantic to the Pamir since the Lias. Tectonoph., v. 123, p. 241-315.

KÖBER, L., 1929 - Die Grossgliederung der Dinariden. Cen=tralbl. f. Mineral., 2 (B), Stuttgart.

LUKOVIĆ, M. & PETKOVIĆ, K., 1952 - Analiza dosadasnjih rado=va i novi podaci o geoloskom sastavu i tektonshom sklopu terena u autohtonoj zoni okoline Ulcinja (Crna Gora). SANU, Pos. izd. knj. CXCVII, Geol. Inst. knj. 4, Beograd.

MARTINI, E., 1971 - Standard Tertiary and Quaternary calcareous nannoplancton zonation. Proceed. II Planktonic Conference, v. 2.

OKADA, H. & BUKRY, D., 1980 - Supplementary modification and introduction of code numbers to the low-latitude Coccolith biostratigraphic zonation (BUKRY, 1973; 1975). Marine Micro=paleontology, v. 5.

OGNIBEN, L., PAROTTO, M. & PRATURLON, A. (Eds.), 1975 - Structural model of Italy. CNR, Quad. Ric. Scient. n. 90, p. 502, Roma.

PAPP, A. & AMSEL, V., 1961 - Uber neue Fossilfunde aus der Bohrung Ulcinj-6, un sudlichen Crna Gora (Montenegro). Geol. Vjesnik, v. 14, Zagreb.

PAVIĆ, A., 1970 - Marinski Paleogen Crne Gore. Stratigra=fija, Tektonika, Paleogeografija. Zav. za Geol. Istr., Titograd.

PERCH-NIELSEN, K., 1985 - Cenozoic calcareous nannofossils. In: BOLLI, SAUNDERS & PERCH-NIELSEN (Eds.), Plankton Stra=tigraphy. Cambridge Univ. Press.

RADOIČIĆ, R. & D'ARGENIO, B., 1988 - Geology and Mesozoic - Early Tertiary facies of External Dinarids (Southern Sector). Guida alle escursioni 74° Congr. Naz. Soc. Geol. It., Sorrento.

ROKSANDIĆ, M. & ČANOVIĆ, M., 1971 - Novi podaci o tektonsko=ne sklopu ulcinkskog producija (New informations about tectonics within the Ulcinj area). Nafta, v. 22/4-5, p. 287-293, Zagreb.

VIDOVIĆ, M., 1958 - Tektonska struktura okoline Ulcinja (Crna Gora). Geol. Glasnik, v. 2, Titograd.

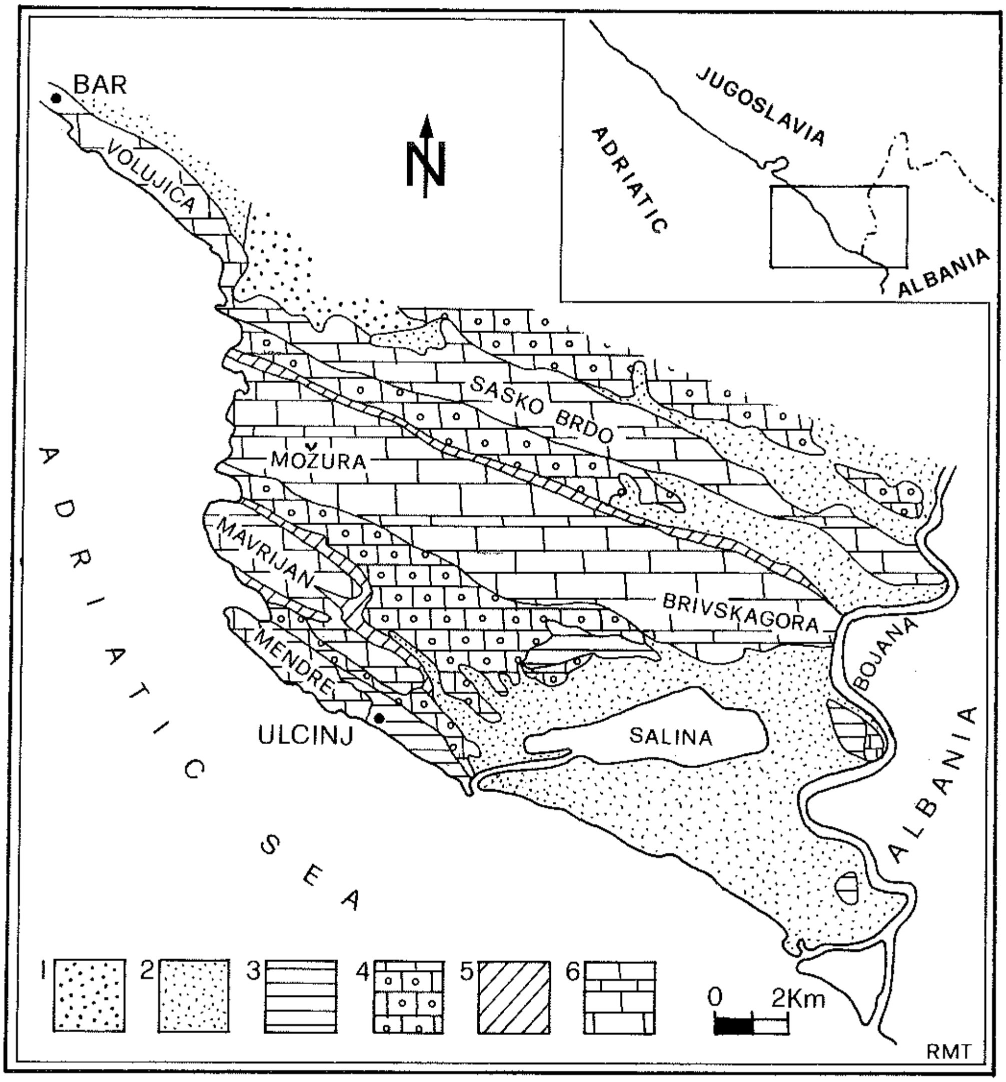

Fig. 1. Schematic geologic map of the Ulcinj-Bar area (after BURIĆ, 1966).
Key (according RADOIČIĆ & D'ARGENIO, 1988): 1. Continental deposits, Holocene – Late PLeistocene; 2. shallow marine to continental deposits, Pleistocene; 3. terrigenous deposits, Miocene; 4. "pelagic" carbonate deposits, Oligocene – Upper Eocene; 5. Open shelf carbonate deposits, Eocene-Paleocene; 6. carbonate platform deposits, Upper Cretaceous.

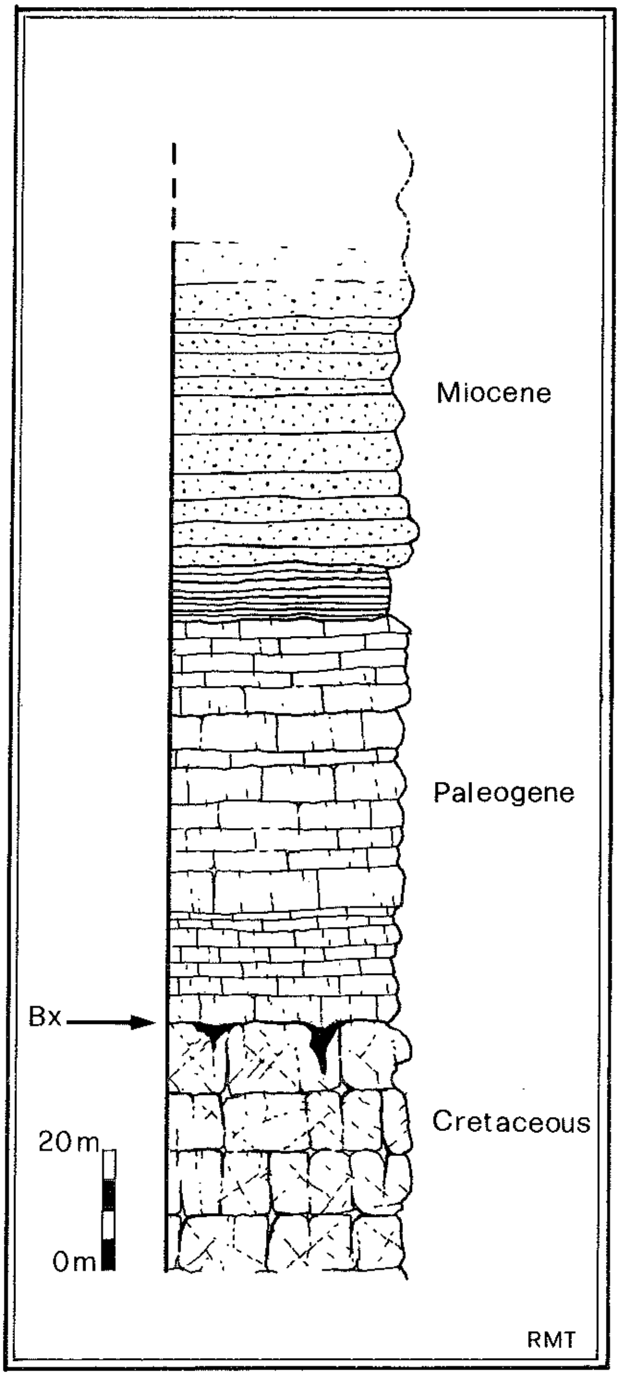

Fig. 2. Columnar section of the northern limb of the Možura anticline.

PLATES

PLATE 1

Fig. 1: *Discoaster tani* BRAMLETTE & RIEDEL, sample 025801.
Fig. 2: *Discoaster nodifer* (BRAMLETTE & RIEDEL) BUKRY, sample 025801.
Figg. 3-4: *Isthmolithus recurvus* DEFLANDRE, sample 025801.
Fig. 5: *Isthmolithus recurvus* DEFLANDRE, sample 025802.
Fig. 6: *Cyclicargolithus floridanus* (ROTH & HAM) BUKRY, sample 025802.
Figg. 7-9: *Sphenolithus pseudoradians* BRAMLETTE & WILCOXON, sample 025801.
Figg. 10-11: *Reticulofenestra oamaruensis* (DEFLANDRE & FERT) STRAANER, sample 025801.
Fig. 12: *Chiasmolithus oamaruensis* (DEFLANDRE) HAY, MOHLER & WADE, sample 025801.
Fig. 13-14: *Pedinocyclus larvalis* (BUKRY & BRAMLETTE) LOEBLICH & TAPPAN, sample 025801.

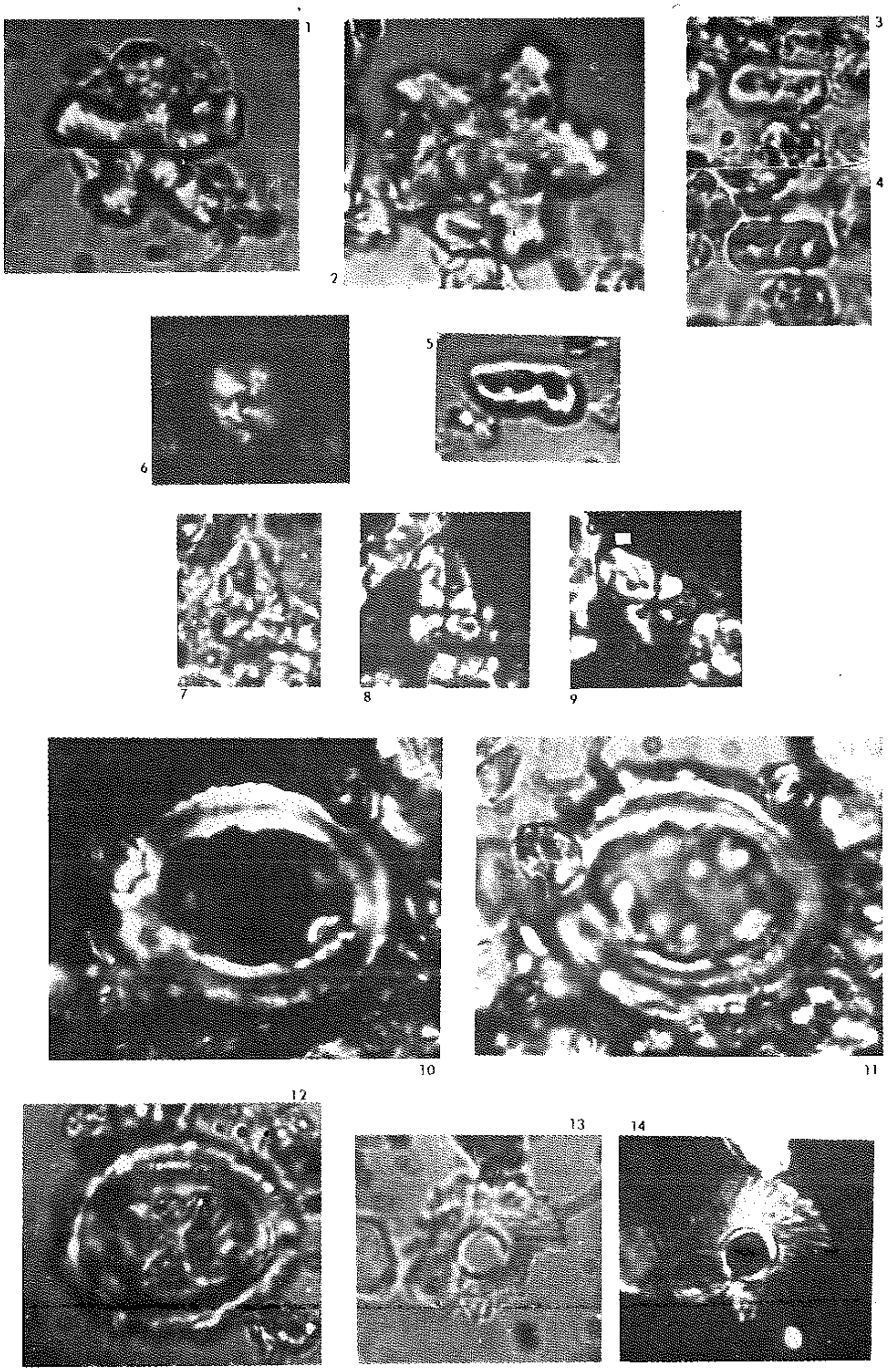
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PLATE 2

Figg. 1-3: *Helicosphaera euphratis* HAQ, sample 025802.
Figg. 4-5: *Reticulofenestra hillae* BUKRY & PERCIVAL, sample 025802.
Figg. 6-7: *Cyclicargolithus abisectus* (MÜLLER) WISE, sample 025804.
Fig. 8: *Discoaster calculosus* BUKRY, sample 025803.
Fig. 9: *Discoaster* cf. *calculosus* BUKRY, sample 025804.
Fig. 10: *Discoaster adamanteus* BRAMLETTE & WILCOKON, sample 025803.
Figg. 11-12: *Ericsonia subdisticha* PERCH-NIELSEN, sample 025802.
Figg. 13-14: *Ericsonia subdisticha* PERCH-NIELSEN, sample 025801.
Fig. 15: *Ericsonia obruta* PERCH-NIELSEN, sample 025802.

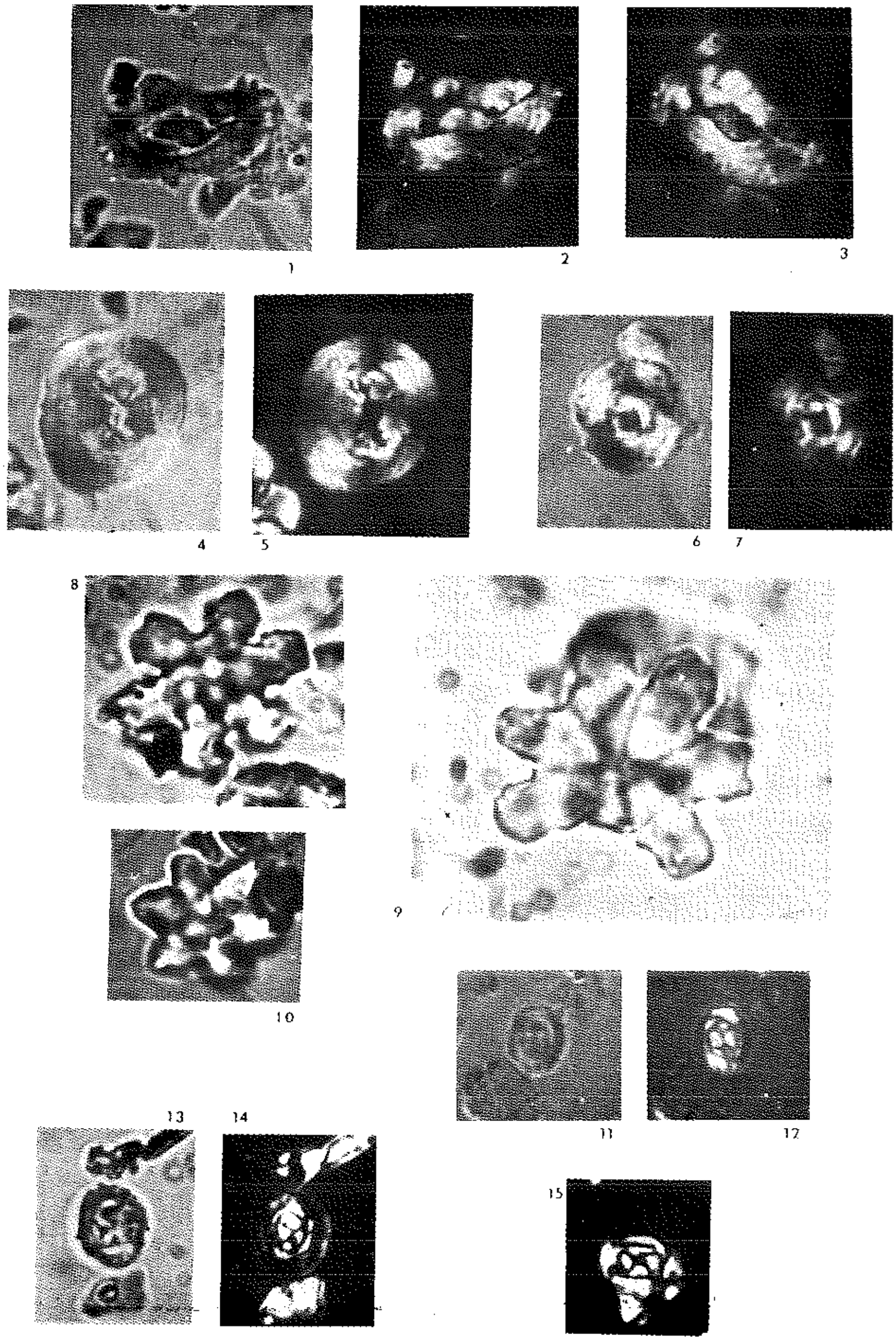
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PLATE 3

Fig. 1: *Discoaster druggi* BRAMLETTE & WILCOXON, sample 025807.
Fig. 2: *Discoaster formosus* MARTINI & WORSLEY, sample 025807.
Fig. 3: *Discoaster kugleri* MARTINI & BRAMLETTE, sample 025804.
Figg. 4-6: *Discoaster bollii* MARTINI & BRAMLETTE, sample 025804.
Figg. 7-8: *Minylitha convallis* BUKRY, sample 025805.
Fig. 9: *Discoaster calcaris* GARTNER, sample 025807.
Figg. 10-11: *Discoaster* cf. *mendomobensis* WISE, sample 025807.
Figg. 12-13: *Sphenolithus heteromorphus* DEFLANDRE, sample 025807.
Figg. 14-15: *Sphenolithus grandis* HAQ & BERGGREN, sample 025806.
Fig. 16: *Sphenolithus abies* DEFLANDRE & FERT, sample 025804.
Figg. 17-18: *Triquetrorhabdulus milowii* BUKRY, sample 025806.
Figg. 19-20: *Triquetrorhabdulus martinii* GARTNER, sample 025805.

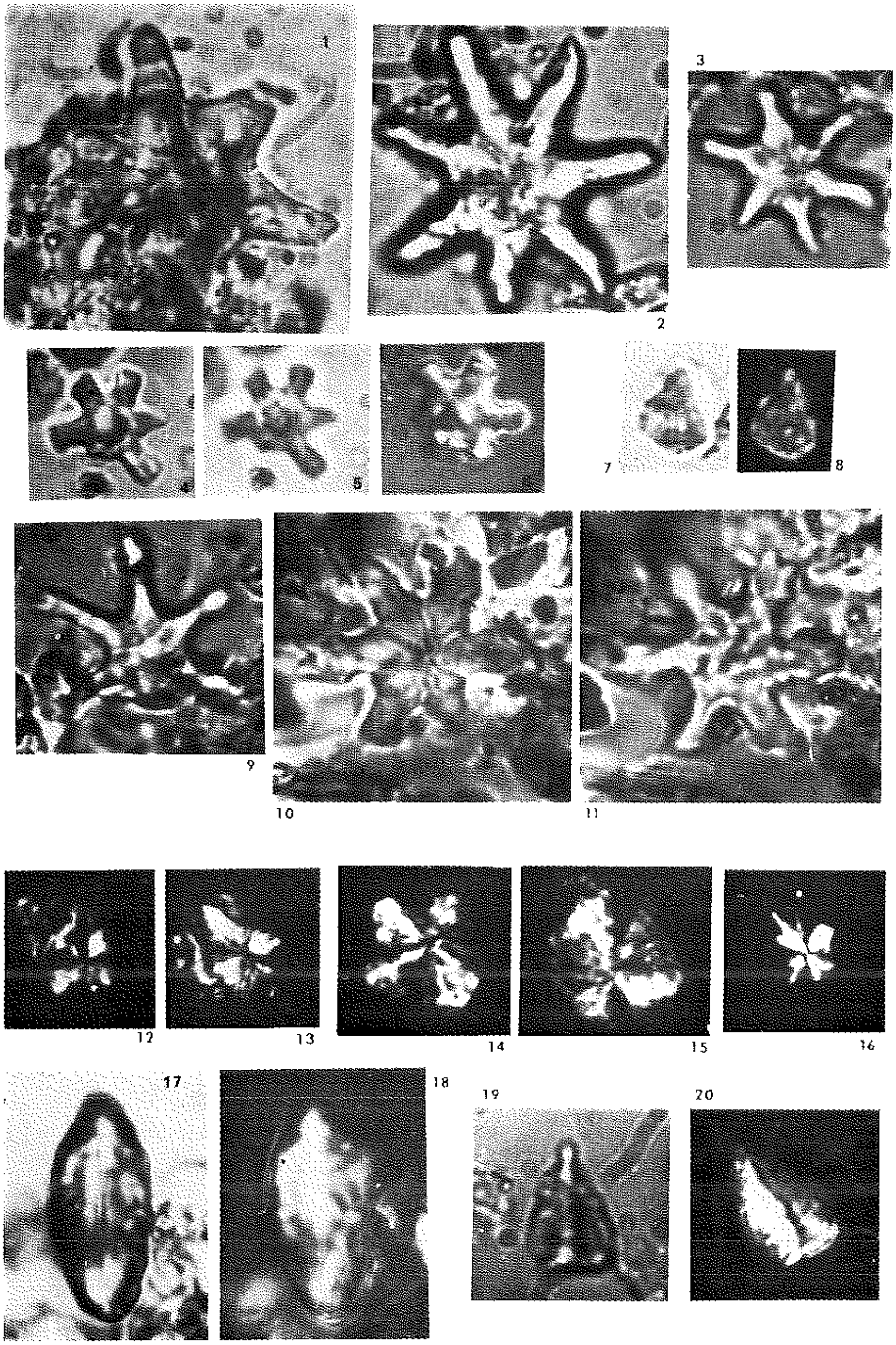
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PLATE 4

Figg. 1-2: *Helicosphaera recta* HAQ, sample 025806.

Figg. 3-5: *Helicosphaera obliqua* (BRAMLETTE & WILCOXON) HAQ, sample 025806.

Figg. 6-7: *Helicosphaera truempyi* BIOLZI & PERCH-NIELSEN, sample 025804.

Fig. 8: *Helicosphaera truempyi* BIOLZI & PERCH-NIELSEN, sample 025806.

Figg. 9-10: *Coccolithus miopelagicus* BUKRY, sample 025807.

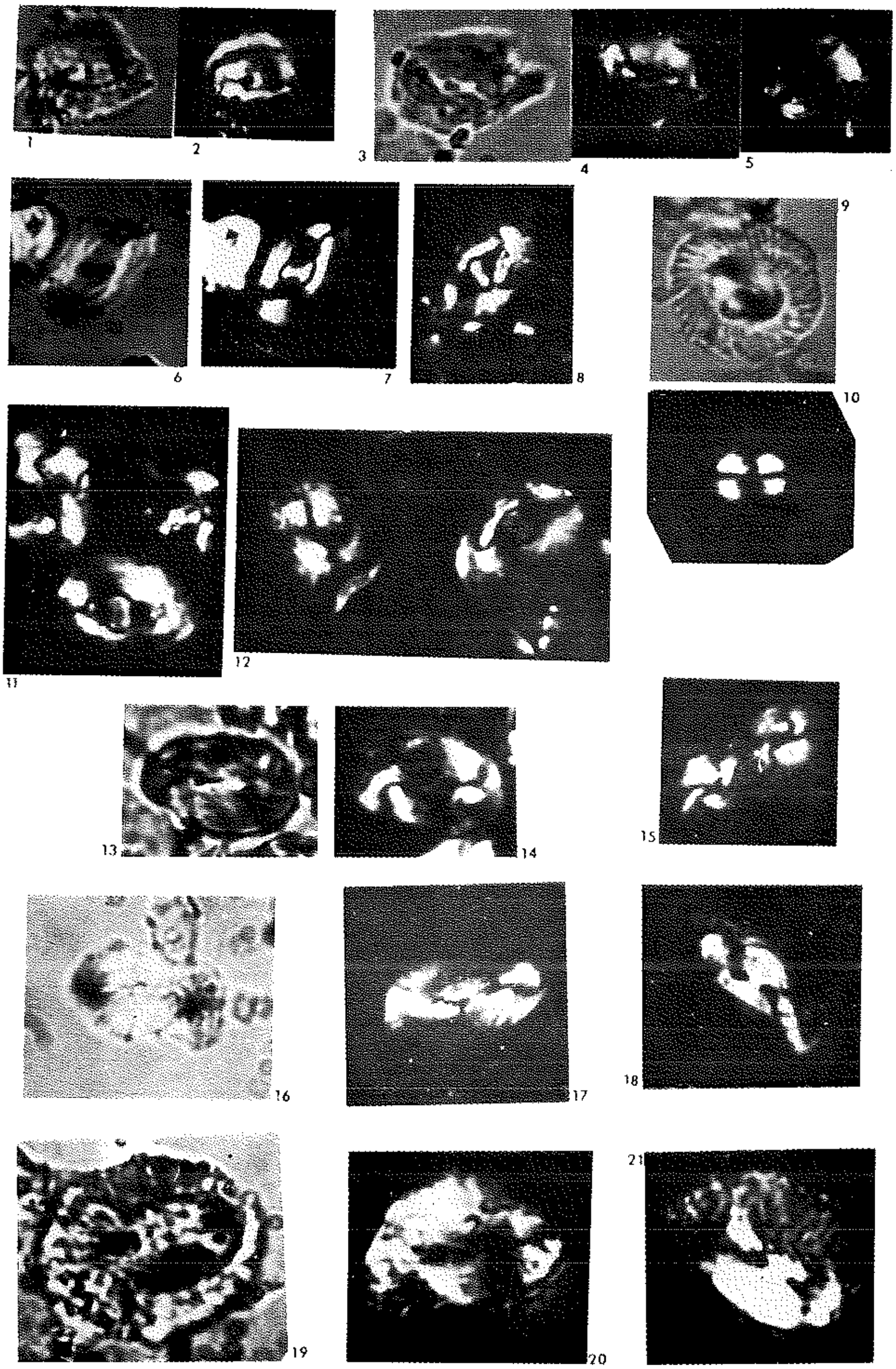
Figg. 11-12: *Helicosphaera truempyi* BIOLZI & PERCH-NIELSEN, sample 025803.

Figg. 13-14: *Helicosphaera scissura* MILLER, sample 025807.

Fig. 15: *Dictyococcites antarcticus* HAQ, sample 025805.

Figg. 16-18: *Helicosphaera carteri* WALLICH, sample 025804.

Figg. 19-21: *Helicosphaera gertae* BUKRY, sample 025806.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INFLUENZA DELLE AGGIUNTE ATTIVE SULL'ALCALI-REAZIONE E SULLA RESISTENZA A COMPRESSIONE DEL RISULTANTE CEMENTO

Nota di Riccardo Sersale e Giuseppe Frigione
Presentata dal Socio Prof. Riccardo Sersale
Adunanza del 3/06/89

__Riassunto__ Gli Autori danno conto dei risultati di una ricerca, rivolta alla possibile correlazione della riduzione di espansione - che fa seguito alla reazione degli alcali con alcuni costituenti reattivi, silicei, talvolta presenti negli aggregati per calcestruzzo di cemento - con i valori attesi di resistenza a compressione. Viene confermato che la scelta del tipo di cemento, in relazione alla natura ed al quantitativo di aggiunta stessa, deve esser di volta in volta commisurato al comportamento meccanico atteso, alle caratteristiche della struttura da realizzare ed al suo ambiente di esercizio.

__Abstract__ The Authors give an account of an investigation devoted to the imperative correlation of the expansion minimization - due to the reaction of the alkalies with some siliceous reactive constituents, sometimes present in the aggregates for cement concrete - with the expected values of compressive strength. It has been confirmed that owing to the different behaviour of each addition in the blend, with reference to its nature and amount, the expected mechanical strengths must always be proportioned to the type of cement selected, as well as to the characteristics of the structure and to environmental conditions.

1. __INTRODUZIONE__

In una precedente Nota (1) abbiamo dato conto dei risultati di una ricerca sperimentale, rivolta a determinare l'influenza delle aggiunte attive di differente natura sull'abbattimento dell'espansione prodotta dalla reazione degli alcali con alcuni costituenti reattivi degli aggregati per calcestruzzo di cemento, interpretandone il meccanismo anche al lume delle risultanze in letteratura.

A completamento ed ampliamento delle precedenti indagini, diamo conto di alcuni risultati, da noi più recentemente conseguiti, inquadrandoli altresì negli ulteriori sviluppi della tematica.

Una ricerca (2) rivolta a determinare, su barre di malta e prismi di calcestruzzo, gli effetti dell'aggiunta di cenere volante, fumo di silice e scoria d'altoforno granulata nella reazione fra gli alcali e tre differenti tipi di aggregato scelti con il proposito di favorire le reazioni: alcali - silice, alcali - carbonato ed alcali - silicato, rispettivamente, ha consentito di avanzare l'ipotesi che il contenuto d'alcali solubili abbia, sì, un ruolo fondamentale, ma che altri parametri, quali il quantitativo di $Ca(OH)_2$ disponibile, la permeabilità delle paste ed il tipo di reazione alcali - aggregato siano da tenere anche in conto. Il ruolo e l'efficacia dell'aggiunta varierebbe infatti in funzione del tipo di alcali reazione.

Una ricerca dedicata all'influenza del fumo di silice (3) ha posto in luce la formazione di idrosilicati microcristallini contenenti alcali, molto simili al tipico gelo alcali - silice. Le proprietà espansive di questi idrosilicati sembrano strettamente collegate al loro contenuto di potassio. Con tutta probabilità, in presenza di fumo di silice, la formazione locale di un gelo calce - alcali - silice non è dannosa, poiché si produce nello spazio disponibile della pasta e il suo sviluppo in forma espansiva avverrebbe molto rapidamente, quando la pasta di cemento, ancora sufficientemente plastica, può sopportare l'aumento di volume senza che insorgano tensioni interne. Le caratteristiche morfologiche di un simile gelo sembrerebbero influenzate oltre che dal contenuto di potassio, anche dal tempo e dalla temperatura d'idratazione.

Uno studio sistematico sull'influenza delle ceneri volanti (4) ha mostrato il benefico effetto di questi materiali nella riduzione dell'espansione. A seguito della partecipazione delle particelle di cenere alla reazione pozzolanica si formerebbe un composto cristallino, stabile, non espansivo grazie al suo alto contenuto di calcio. La sostituzione di un'aliquota di cemento con ceneri volanti promuoverebbe indirettamente un decremento del contenuto di alcali della soluzione nei pori, sicché soltanto gli alcali solubili, e non quelli totali del conglomerato, avrebbero un ruolo predominante nell'alcali reazione.

Una ricerca dedicata all'influenza della scoria granulata (5), sia come aggiunta attiva, sia come aggregato, ha posto in luce che l'inibizione dell'espansione non é semplicemente addebitabile alla diluizione degli alcali del clinker, ma ad una specifica azione della scoria. Nello studio della reazione dell'idrossido di sodio con la scoria e con le ceneri volanti, rispettivamente, sono state infatti riscontrate differenze di comportamento, in quanto mentre le seconde cedono abbondantemente silice alla soluzione, riducendo l'alcalinità a seguito del consumo di alcali, la scoria rilascia quantità molto piccole di silice e, quindi, riduce di poco l'alcalinità. L'alcali della soluzione originaria tende, pertanto, ad essere incorporato nella pasta idratata. E' noto, infatti, che l'alcali della scoria si scioglie in quantità molto piccola anche in soluzione di idrossido di calcio.

Se è ormai accertato che la migliore misura preventiva per la riduzione dell'espansione che accompagna l'alcali reazione è rappresentata dall'addizione al clinker Portland di aggiunte attive di varia natura e se sul meccanismo che regola l'abbattimento dell'espansione sono riportare ragionevoli e valide interpretazioni, anche nella più recente letteratura, non altrettanto si verifica sotto il profilo della correlazione fra natura e quantitativo di aggiunta attiva, da un lato, e comportamento meccanico del risultante cemento, dall'altro. Abbiamo perciò creduto interessante apportare un contributo in tale direzione, effettuando una ricerca sperimentale rivolta a correlare la riduzione di espansione, conseguibile con l'impiego di aggiunte attive di differente natura, addizionate in quantità differenti, ai valori di resistenza a compressione su malta standard del corrispondente cemento.

Delle modalità sperimentali e delle relative conclusioni riferiamo nel corso di questa Nota.

2. PARTE SPERIMENTALE

Impiegando sempre lo stesso clinker industriale, a contenuto di Na_2O equiv. pari ad 1,32 e macinato a superficie specifica Blaine di 380 m^2/kg, sono stati preparati cementi di miscela contenenti quali aggiunte attive, in proporzione compresa fra il 10 ed il 40%, alcuni campioni di piroclastiti flegree, laziali e vulsine, di ceneri volanti (residui della combustione di polverino di carbone per la produzione di energia elettrica), un campione di tufo giallo napoletano, un campione di marna silicea (Pavia) ed alcuni campioni di scoria d'altoforno granulata, quest'ultima in proporzione compresa fra il 10 e l' 80%. Le aggiunte a comportamento pozzolanico sono state macinate fino a lasciare un residuo del 10% sul setaccio da 40 μm; i campioni di scoria sono stati portati alla stessa finezza del clinker.

Ai fini della valutazione della reattività potenziale verso gli alcali dei cementi di aggiunta ottenuti, si è fatto ricorso al disposto della Norma ASTM Standard C 441 - 81 che prevede la misura dell'espansione su barre di malta confezionate impiegando quale aggregato reattivo il vetro Pyrex. Per le modalità di stagionatura delle malte, protratta fino a due anni, si è seguita la Norma ASTM Standard C 227 - 81.

Le Figg. 1, 2, 3 riportano i risultati delle prove di espansione sulle barre di malta, preparate ciascuna con i differenti cementi di miscela contenenti quali aggiunte attive i prodotti a comportamento pozzolanico. Può rilevarsi il livello ottimale di sostituzione che si manifesta compreso fra il 30 ed il 40% ed il fatto che l'abbattimento dell'espansione é notevole, ma non risulta uguale per opera di tutti i prodotti naturali. Si registrano, infatti, diffe-

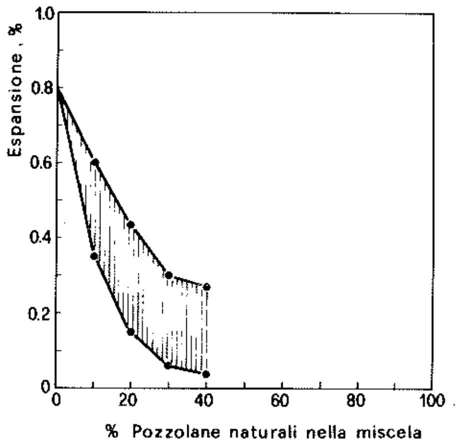


Fig. 1. Sostituzione parziale di clinker Portland con pozzolane naturali. Effetto sull'espansione.

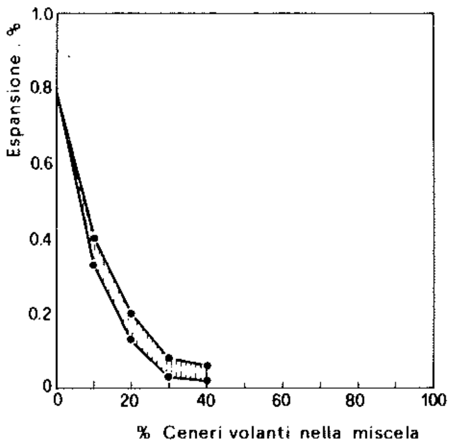


Fig. 2. Sostituzione parziale di clinker Portland con ceneri volanti. Effetto sull'espansione.

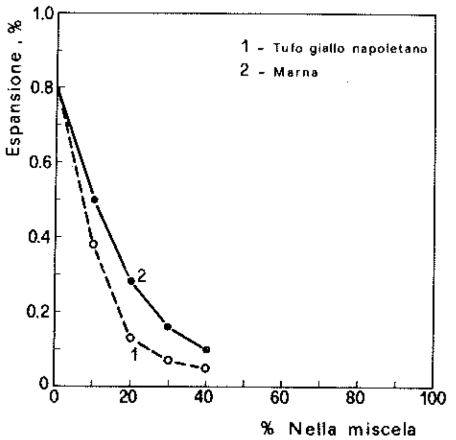


Fig. 3. Sostituzione parziale di clinker Portland con tufo giallo napoletano o marna silicea. Effetto sull'espansione.

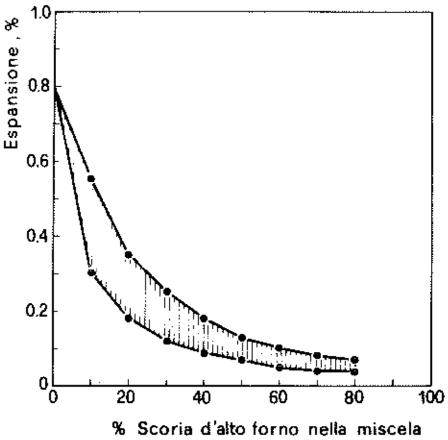


Fig. 4. Sostituzione parziale di clinker Portland con scorie d'altoforno granulate. Effetto sull'espansione.

renze sensibili fra pozzolana e pozzolana e si conferma il miglior comportamento dell'equivalente zeolitizzato della pozzolana flegrea: il tufo giallo napoletano (6). Ceneri volanti ed ancor più scorie d'altoforno, Fig. 4, denunziano invece un comportamento più simile in seno al cemento.

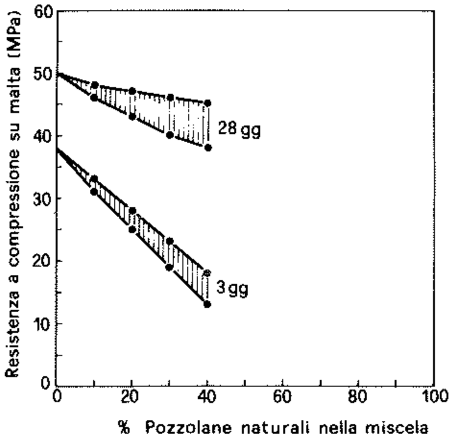
La Fig. 4 consente di rilevare, inoltre, che abbattimenti dell'espansione dell'ordine di quelli conseguibili con i cementi contenenti prodotti a comportamento pozzolanico, sono ottenibili con cemento d'altoforno a livelli di sostituzione del clinker ben maggiori e pari a circa il 60% di scoria granulata.

Le Figg. 5, 6, 7 ed 8 riportano i valori rilevati dalle prove di resistenza a compressione su malte ISO - RILEM (7) preparate con ciascuno dei cementi di aggiunta in esame e stagionate 3 e 28 giorni. Si rileva che, per livelli di sostituzione del clinker necessari ad abbattere significativamente l'espansione da alcali reazione, alle brevi stagionature le aggiunte di prodotti a comportamento pozzolanico forniscono cementi con valori di resistenza a compressione ben più alti di quelli dei cementi contenenti scorie d'altoforno. Alle lunghe stagionature si registra l'inverso: i valori di resistenza a compressione forniti dai cementi d'altoforno superano di circa 10 MPa quelli rilevati con i cementi contenenti prodotti a comportamento pozzolanico.

Misure di lavorabilità effettuate mediante prova di spandimento (8) hanno confermato, a parità di contenuto d'acqua, una migliore lavorabilità, nell'ordine, dei calcestruzzi confezionati con cementi contenenti scoria d'altoforno e con cementi contenenti ceneri volanti. I prodotti naturali a comportamento pozzolanico, richiedendo una maggiore quantità d'acqua d'impasto per una buona lavorabilità, sottolineano l'opportunità dell'impiego di additivi convenientemente scelti, per contrastare l'inevitabile abbassamento di resistenza meccanica.

3. CONCLUSIONI

L'insieme delle risultanze sperimentali conferma la notevole influenza delle aggiunte attive a comportamento poz-



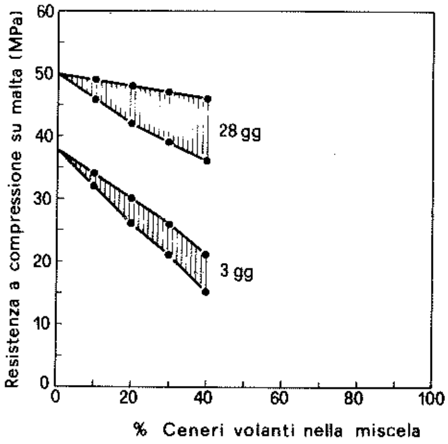
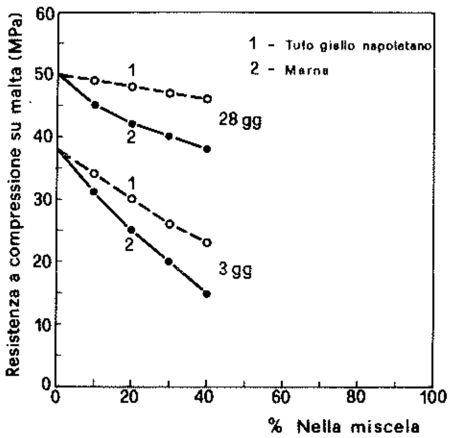


Fig. 5. Sostituzione parziale di clinker Portland con pozzolane naturali. Effetto sulla resistenza a compressione.

Fig. 6. Sostituzione parziale di clinker Portland con ceneri volanti. Effetto sulla resistenza a compressione.



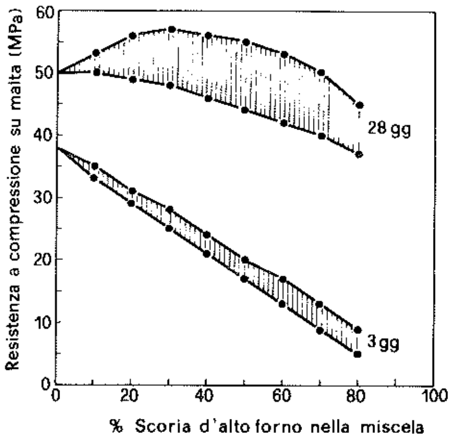


Fig. 7. Sostituzione parziale di clinker Portland con tufo giallo napoletano o marna silicea. Effetto sulla resistenza a compressione.

Fig. 8. Sostituzione parziale di clinker Portland con scorie d'altoforno granulate. Effetto sulla resistenza a compressione.

zolanico, a livelli di sostituzione del clinker compresi fra il 20 ed il 40%, sull'abbattimento dell'espansione per alcali reazione e mostra che, se il meccanismo d'azione può anche esser diverso in funzione del tipo di aggiunta, il risultato è abbastanza simile.

Va però tenuto conto del fatto che se l'abbattimento dell'espansione vuole esser ottenuto con l'impiego di cementi d'altoforno, valori simili a quelli raggiungibili con aggiunte di prodotti a comportamento pozzolanico a livelli di sostituzione del clinker dell'ordine del 30%, sono ottenibili con l'aggiunta di una quantità di scoria granulata ben maggiore e pari a circa il doppio. Sicchè la scelta del tipo di cemento di miscela va fatta indubbiamente in funzione del tipo di opera da realizzare e del relativo ambiente di vita, ma altresì dopo attenta correlazione fra abbattimento dell'espansione da alcali reazione e resistenza meccanica attesa.

Va altresì sottolineato che, nel caso d'impiego di cementi di miscela preparati con pozzolane naturali, si possono registrare differenze nei valori attesi di abbattimento dell'espansione, differenze che sono attribuibili al tipo di piroclastite impiegata.

4. BIBLIOGRAFIA

(1) Sersale R., Frigione G.; "On the minimization of the alkali-aggregate reaction. Influence of pozzolanas, powdered volcanic tuff, blast furnace slag and fly ash". 7th International Conference on Alkali-Aggregate Reaction. Poster Sect., Ottawa, Canada (1986).

(2) Durand B., Berard J., Soles J.A.; "Comparison of the effectiveness of four mineral admixtures to counteract alkali-aggregate reaction". Proc. 7th International Conference on Alkali-Aggregate Reaction. p 30-35. Ottawa, Canada (1986).

(3) Tenoutasse N., Marion A.M.; "The influence of silica fume in alkali-aggregate reactions". Proc. 7th

International Conference on Alkali-Aggregate Reaction. p. 71-75. Ottawa, Canada (1986).

(4) Tenoutasse N., Marion A.M.; "Influence of fly ash in alkali-aggregate reaction". Proc. 7th International Conference on Alkali-Aggregate Reaction. p. 44-48. Ottawa, Canada (1986).

(5) Yamamoto C., Makita M., Moriyama Y., Numata S.; "Effect of ground granulated blast furnace slag admixture and granulated or air-cooled blast furnace slag aggregate on alkali-aggregate reactions and their mechanisms". Proc. 7th International Conference on Alkali-Aggregate Reaction. p. 49-52. Ottawa, Canada (1986).

(6) Sersale R., Frigione G.; "Blended cements manufactured with activated zeolitic additions". Chimica e Industria, 10, p 537-541 (1985).

(7) ISO Recommendation R. 679 (1968).

(8) DIN 1048.

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AN ALGORITHMIC APPROACH TO IDEAL GENERATION OF POINTS.

Nota di Isabella Ramella *
Presentata dal socio Prof. Ciro Ciliberto
Adunanza del 4/11/89

Riassunto. Si costruiscono algoritmi per calcolare il numero e i gradi dei generatori di un ideale di punti dello spazio proiettivo.
Abstract. On construct algorithms for computing the number and degrees of generators of an ideal of points in the projective space.

INTRODUCTION.

Let $V=\{P_1,...,P_s\}$ be a set of points of the projective space $\mathbb{P}^r_k$ (over any field k) and $I=I(V)\subset k[X_0,....,X_r]$ be the ideal of V. Let $f_1,f_2,...,f_\nu$ be a minimal set of homogeneous generators of V. In this paper, starting from the coordinates of the points P_i, we construct an algorithm for computing ν and the degrees, deg f_i, of the polynomials f_i. If the points do not lie on the hyperplane $x_0=0$ (if k is infinite, we can always reduce to this case by a change of coordinates) the algorithm can be improved. If the points are in "general position" and $k=\mathbb{Q}$ we show that the computations can be reduced to the case $k=\mathbb{Z}p$, thus shortening the computational time.We have implemented the algorithm, in the case $k=\mathbb{Q}$, using the computer system Macsyma on a Vax station 2000 with VMS operating system and, in the case $k=\mathbb{Z}p$, using a Basic compiler on an IBM PC (but in the case $k=\mathbb{Z}p$, pratically any language and any computer works). This paper has been inspired by a computational proof of the Ideal Generation Conjecture for 28 points in $\mathbb{P}^3_k$, given in **[R]**. The computational aspect of this work will be part of a Ph.D. thesis in course of preparation.

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1. GENERALITIES.

Let $S = \bigoplus_{d\in\mathbb{N}} S_d$ be a graded Cohen-Macaulay finitely generated k-algebra (S_0=k, k any field).Then, if$\{x_0, ..., x_r\}$ is a basis of the k-vector space S_1, $S=k[x_0, ..., x_r]=R/I$ where $R = k[X_0, ..., X_r]$. We have $R = \bigoplus_{d\in\mathbb{N}} R_d$, $I= \bigoplus_{d\in\mathbb{N}} I_d$ and $S_d=R_d/I_d$.

Let dim S = δ. The Hilbert function of S is defined by $H(S,d) = \dim_k S_d$. Clearly :

(1.1) $H(S,d) = \dim_k R_d - \dim_k I_d = \binom{d+r}{r} - \dim_k I_d$.

There is a numerical polynomial $P(S,Z)\in\mathbb{Q}[Z]$ of degree δ-1, the Hilbert polynomial of S, such that H(S,d) = P(S,d) for $d>>0$.

Following the usual notations in the literature, throughout this paper we set:

(1.2) $\alpha = \text{Min}\ \{d|I_d\neq 0\}$ and $\sigma=\text{Min}\{d|H(S,d)=P(S,d)\}+\delta$.

In the sequel when we write $I=(f_1, ...,f_\nu)$ we imply that $(f_1, ...,f_\nu)$ is a minimal set of homogeneous generators of I, with $\deg f_i \leq \deg f_{i+1}$, $i = 1, ..., \nu$.

Proposition 1.3 . If $I=(f_1, ...,f_\nu)$, then $\deg f_i = \alpha$ and $\deg f_\nu \leq \sigma$.

Proof. The equality is trivial.The inequality is proved in [**DGM** , Proposition 3.7].

The following is an easy consequence of Nakayama's Lemma. Two minimal set of homogeneous generators of I have the same number of elements in any degree,that is:

(1.4) if ν_d is the number of elements f_i of degree d, the integers ν and ν_d are invariants of the ideal I.

From now on we set :

$\nu(I)=\nu$ and $\nu_d(I)=\nu_d$

clearly $\nu = \sum_{d=\alpha}^{\sigma} = \nu_d$

Let W_d be the following subvector space of I_d:

(1.5) $W_d = X_0 I_{d-1}+...+X_r I_{d-1}$ (W_d=0 if $d\leq\alpha$).

We have:

(1.6) $v_d(I)=\dim_k I_d-\dim_k W_d \quad \alpha\le d\le\sigma.$

Suppose now that x_0 is a non zero divisor of S (if k is infinite there is always a non zero divisor in S_1 and, by a change of coordinates, we can assume, with no loss of generality, that it is x_0).

If $f\in k[X_0,\dots,X_r]=R$, we denote with $\bar{f}\in k[X_0,\dots,X_r] / (X_0)=K[X_1,\dots,X_r]$ the polynomial obtained by f deleting all monomials wich involve X_0 ; if U is a set of polynomials $f\in R$, $\overline{U}=\{\bar{f}|f\in U\}\subset K[X_1,\dots,X_r]$.

Clearly $\overline{W}_d=X_1\bar{I}_{d-1}+\dots+X_v\bar{I}_{d-1}$. If $I=(f_1,\dots f_v)$ the following natural isomorphism holds:

$$S/(x_0)\cong R/(I+(X_0))\cong K[X_1,\dots,X_r]/\bar{I}$$

where $\bar{I}$ is the ideal generated by $\bar{f}_1,\dots,\bar{f}_v$.

Since $S/(x_0)\cong K[X_1,\dots,X_r] / \bar{I}$ is a Cohen Macaulay graded k-algebra we can apply all the previous considerations to the ideal $\bar{I}$.

Lemma 1.7 For any $d\in\mathbb{N}$, we have:

a) $v(I)=v(\bar{I})$ and $v_d(I)= v_d(\bar{I})$,

b) $\dim_k \bar{I}_d= \dim_k I_d - \dim_k I_{d-1}$,

c) $v_d(I)= \dim_k I_d - \dim_k I_{d-1} - \dim_k\overline{W}d$.

Proof. a) The ideal J is generated by $\bar{f}_1,\dots,\bar{f}_v$ and we have only to prove that $\bar{f}_1,\dots,\bar{f}_v$ is minimal. Suppose that $J = (\bar{f}_1,\dots,\bar{f}_{i-1}, \bar{f}_{i+1},\dots,\bar{f}_v)$ then $\bar{f}_i = \sum_{j=1}^{i-1} \bar{g}_j \bar{f}_j$

and $f_i - \sum_{j=1}^{i-1} g_j f_j = X_0h\in I$, where $\deg h<\deg f_i$. But, if x_0 is a non zero divisor of S, $h\in I$ and then $h = \sum_{j=1}^{i-1} g'_j f_j$.

Thus $f_i = \sum_{j-1}^{i-1} (g_j +X_0g'_j) f_j$ against the assumption of minimality of $\{f_1,\dots f_v\}$.

b) We have $\bar{I}= I+(X_0)/(X_0) \cong I/(X_0) \cap I$. But $(X_0) \cap I=X_0I$ since X_0 is a non zero divisor of S. Then $\bar{I}\cong I/X_0I$ and $\bar{I}_d\cong I_d/X_0I_{d-1}$.

Now , since x_0 is a non-zero divisor of S, $\dim_k I_{d-1}=\dim_k(X_0 I_{d-1})$.Then the claim.

c) Follows immediately from a) and b).

2. AN ALGORITHM FOR COMPUTING v(I) and deg f_i WHEN I IS THE IDEAL OF POINTS OF $\mathbb{P}^r_k$.

In this section we consider the algebraic variety consisting of a finite set of points $V=\{P_1,\dots,P_s\}\subset\mathbb{P}^r_k$ and the ideal of V, $I=I(V)=\{f\in R \mid f(P_i)=0, i=1,\dots,s\}$.

Let S=R/I be the homogeneous coordinate ring of V.

From elementary commutative algebra we have :

(2.1) The zero divisors of S, are the elements of $\bigcup_{i=1}^{s} \mathfrak{p}_i$, where $\mathfrak{p}_i$ are the minimal primes of S (corresponding to the points P_i). Then there is always a non zero divisor in S, that is S is a graded Cohen-Macaulay finitely generated k-algebra. Thus we can apply to S the results of Section 1. The vector space I_d is easily described as the null space of a matrix with elements in k. In fact:

(2.2) $I_d = \{f\in R_d \mid f(P_i)=0 \text{ for } i=1,..,s\}$

If we denote with T_i, $i=1,\dots,N(d)=\binom{d+r}{r}$, the terms of degree d in the indeterminates $X_0,\dots,X_r$ then $\{T_i\}_{i=1,\dots,N(d)}$ is a basis of the k-vector space R_d. Let G^s_d be the sxN(d) matrix :

(2.3) $G^s_d=(t_{ij})$ where $t_{ij}=T_i(P_j)$.

If $f=\sum_{i=1}^{N(d)} \lambda_i T_i\in R$, we set:

$$G^s_d\, f= G^s_d \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{N(d)} \end{pmatrix}$$

Then:

(2.4) $I_d=\{f\in R_d \mid G^s_d\, f=0\}$, that is I_d is the <u>null space</u> of the matrix G^s_d.

Hence:

(2.5) $H(S,d)=N(d)-\dim_k I_d=\rho(G_d^s)\leq \mathrm{Min}\{s,N(d)\}$ (ρ=rank), for any $d\in\mathbb{N}$

and

$$\alpha=\mathrm{Min}\{d \mid I_d\neq 0\}=\mathrm{Min}\{d \mid \rho(G_d^s)<N(d)\}$$

(2.6)

$$\sigma=\mathrm{Min}\{d \mid \rho(G_d^s)=s\}+1$$

Let $\{g_1,\ldots,g_h\}$ $(h=\nu_{d-1})$ be a basis of I_{d-1}, then the $h(r+1)$ polynomials g_iX_j of I_d span the vector space W_d (see (1.5)) and if H_d^s is the matrix whose rows are the coordinates of the vectors g_iX_j in R_d, then

(2.7) $$\dim_k W_d = \rho(H_d^s)$$

Collecting all the previous results we have :

(2.8) Algorithm

Input : Coordinates of the points $P_1,\ldots,P_s$ in $\mathbb{P}_k^r$

Output : ν_d= number of elements of degree d of a minimal basis of I_d

ν= number of elements of a minimal basis of I.

Step 1. Set $d=0$, $\alpha=0$, $\sigma=0$.

Step 2. Reset $d=d+1$.

Step 3. If $\sigma=0$ and $\rho(G_d^s)=s$ then set $\sigma=d+1$.

Step 4. If $\alpha=0$ and $\rho(G_d^s)<N(d)$ then set $\alpha=d$ and $\nu_\alpha=N(\alpha)-\rho(G_\alpha^s)$ else return to step 2.

Step 5. Compute a basis of I_d (that is a basis of the null space of G_d^s).

Step 6. Construct the matrix H_{d+1}^s and set $\nu_{d+1}=N(d+1)-\rho(G_{d+1}^s)-\rho(H_{d+1}^s)$.

Step 7. If $d+1=\sigma$ then continue else go to Step 2.

Step 8. Set $\nu=\sum_{d=\alpha}^{\sigma}\nu_d$.

If x_0 is a non zero divisor of S the previous algorithm can be improved. In fact by Lemma 1.7 and (2.5) we have

$$(2.9) \qquad v_d(I)=\binom{d+r-1}{r-1} - \rho(G_d^s) + \rho(G_{d-1}^s) - \dim_k \overline{W}_d.$$

If $\{g_1,\dots,g_h\}$ $(h=v_{d-1})$ is a basis of I_{d-1}, the vector space $\overline{W}_d$ is spanned by the hr polynomials $\overline{g}_j X_j$ $j=1,\dots,r$, and if $\overline{H}_d^s$ is the matrix whose rows are the coordinates of the vectors $\overline{g}_j X_j$ of $\overline{R}_d \subset K[X_1,\dots,X_r]$ we have :

$$(2.10) \qquad \dim_k \overline{W}_d = \rho(\overline{H}_d^s),$$

where the matrix $\overline{H}_d^s$ has size $hr \times \binom{d+r}{r-1}$, which is smaller than the size $h(r+1) \times \binom{d+r+1}{r}$ of H_d^s.

(2.11) If x_0 is a non zero divisor of S, in virtue of (2.9) and (2.10) by substituting, in algorithm (2.8), 6) with

Step 6') construct the matrix $\overline{H}_d^s$ and set:

$$v_{d+1} = \binom{d+r}{r-1} - \rho(G_{d+1}^s) + \rho(G_d^s) - \rho(\overline{H}_{d+1}^s)$$

we obtain another better algorithm for computing v_d and v.

Remark 2.12 By (2.1), x_0 is a non zero divisor of S if and only if the hyperplane $x_0=0$ does not contain any point P_i $i=1,\dots,s$, that is the first coordinate of P_i is not null for any $i=1,\dots,s$. If k is infinite, by a linear change of coordinates, we can always assume that this case.

3. THE CASE OF POINTS IN GENERIC POSITION. THE IDEAL GENERATION CONJECTURE.

If S is the homogeneous coordinate ring of s points $\{P_1,\dots,P_s\}$ of $\mathbb{P}_k^r$, by (2.4), we have $H(S,d)=\rho(G_d^s) \leq \mathrm{Min}\{s,N(d)\}$ for any $d \in \mathbb{N}$.

Definition 3.1. If $H(S,d)=\mathrm{Min}\{s,N(d)\}$, for any $d \in \mathbb{N}$, the points $\{P_1,\dots,P_s\}$ are said to be in generic position. The properties of points in generic position were first investigated in **[0]**. In **[G0]** is proved that almost all set of points are in generic position. In this case Algorithm (2.8) and (2.11) can be simplified since it is possible to do all computations in the degrees α and $\alpha+1$, where $\alpha=\mathrm{Min}\{d \mid I_d \neq 0\}$ (see (1.2)), as we show with the following results.

Lemma 3.2 If $\{P_1,...,P_s\}$ are in generic position, then $\alpha=\text{Min}\{d|s<N(d)\}$ and $\sigma\leq\alpha+1$.

Proof. By (2.6) we have $\sigma-1=\text{Min}\{d|\ H(S,d)=s\}$ and $\alpha=\text{Min}\{d|\ H(S,d)<N(d)\}$. Since, for any d, we have $H(S,d)=\text{Min}\{s,N(d)\}$ then $\alpha=\text{Min}\{d|s<N(d)\}$ and $H(S,\sigma-2)=N(\sigma-2)$.
Thus $\sigma-2<\alpha$, that is the claim.

Corollary 3.3 Let $\gamma=\text{Min}\{d|s<N(d)\}$.

a) If $H(S,\gamma-1)=\rho(G^s_{\gamma-1})=N(\gamma-1)$ and $H(S,\gamma)=\rho(G^s_\gamma)=s$, the points $\{P_1,...,P_s\}$ are in generic position (hence $\gamma=\alpha$, by Lemma 3.2).

b) If $\{P_1,...,P_s\}$ are in generic position then

$$\nu(I)=N(\alpha)+N(\alpha+1)-2s-\rho(H^s_{\alpha+1}).$$

Proof. a) Since $\rho(G^s_d)=N(d)$ for any $d<\gamma-1$, the claim is consequence of the well known fact that if $H(S,\gamma)=s$ then $H(S,d)=s$ for any $d>\gamma$.

b) By Lemma 3.2 and Proposition 1.3 we have $\nu(I)=\nu_\alpha(I)+\nu_{\alpha+1}(I)$ and, by (1.6) and (2.5), $\nu_\alpha(I)=N(\alpha)-\rho(G^s_\alpha)$, $\nu_{\alpha+1}=N(\alpha+1)-\rho(G^s_{\alpha+1})-\rho(H^s_{\alpha+1})$. Finally $\rho(G^s_\alpha)=\rho(G^s_{\alpha+1})=s$, by Definition 3.1, and then the claim.

Remark. Since $\sigma\geq\alpha$, by Lemma 3.2, two cases are possible : $\sigma=\alpha$ or $\sigma=\alpha+1$. In particular, when $H(S,\alpha-1)=N(\alpha-1)<s$, $\sigma=\alpha$. Thus, in this case, we have $\nu(I)=\nu_\alpha(I)=N(\alpha)-s$.

Let $\overline{H}^s_{d+1}$ be the matrix of (2.10).

Corollary 3.4. Let $\alpha=\text{Min}\{d|\ N(d)>s\}$ and $\{P_1,...,P_s\}$ be points in generic position such that their first coordinate is not null. Then :

$$\nu(I)=N(\alpha+1)-s-\rho(\overline{H}^s_{\alpha+1}).$$

Proof. By Lemma 1.7, c) and (2.10) we have:
$\nu_\alpha(I)+\nu_{\alpha+1}(I)=\dim_k I_{\alpha+1}-\dim_k\overline{W}_{\alpha+1}=\dim_k I_{\alpha+1}-\rho(\overline{H}^s_{\alpha+1})=N(\alpha+1)-\rho(G^s_{\alpha+1})-\rho(\overline{H}^s_{\alpha+1})$
and by assumption $\rho(G^s_{\alpha+1})=s$. Then the claim.

By (2.7), $\dim_k W_{\alpha+1}=\rho(H^s_{\alpha+1})\leq \mathrm{Min}\{(\dim_k I_\alpha)(r+1),\dim_k I_{\alpha+1}\}$ and by corollary 3.3, $v(I)\geq N(\alpha)+N(\alpha+1)-2s-\mathrm{Min}\{(\dim_k I_\alpha)(r+1),\dim_k I_{\alpha+1}\}$.

In **[G0]** the following conjecture has been made:

Ideal generation conjecture. Fixed s and r for points $\{P_1,\ldots,P_s\}\subset\mathbb{P}^r_k$ in "general position", $\dim_k W_{\alpha+1}$ is as by as it could possibly be.

This means that v(I) is the least possible :

(3.5) $$v(I)=N(\alpha)+N(\alpha+1)-2s-\mathrm{Min}\{(N(\alpha)-s)(r+1),N(\alpha+1)-s\}.$$

It is easily seen that in order to prove the conjecture it is enough to find, for any r, s an example of points for which (3.5) holds. Various authors have tried to prove the "Ideal Generation conjecture" for any s and r. In $\mathbb{P}^2$ the conjecture has been proved in **[GM]**. In $\mathbb{P}^3$ an inductive proof is given in **[B]**. The starting point of the induction, is the case of 28 points which has been proved in **[H]**. After that a computational proof of this case has been given **[R]**. This last paper has inspired our work. No proof of the conjecture is known for any $r\geq 4$.

By the previous results it is possible to write an algorithm for finding points that satisfy the ideal generation conjecture.

3.6 Algorithm.

Input : coordinates of the points $\{P_1,\ldots,P_s\}\subset\mathbb{P}^r_k$.

Output : $\{P_1,\ldots,P_s\}$ satisfy the I.G.C.

Step 1. Let $\alpha=\mathrm{Min}\{d|N(d)>s\}$. If $\rho(G^s_{\alpha-1})=N(\alpha-1)$ and $\rho(G^s_\alpha)=s$ then $\{P_1,\ldots,P_s\}$ are in generic position else stop.

Step 2. If $N(\alpha-1)=s$ then $\{P_1,\ldots,P_s\}$ satisfy the I.G.C. and stop else continue.

Step 3. Compute a basis $\{g_1,\ldots,g_h\}$ of I_α and construct the matrix $\overline{H}^s_{\alpha+1}$, whose rows are the coordinates of the vector $\overline{g}_iX_j$; if $\rho(\overline{H}^s_{\alpha+1})=\mathrm{Min}\{(N(\alpha)-s)(r+1),N(\alpha+1)-s\}$ then $\{P_1,\ldots,P_s\}$ satisfy the I.G.C., otherwise stop.

As in section 2, if x_0 is a non zero divisor, the previous algorithm can be simplified by substituting the matrix $\overline{H}^s_{\alpha+1}$ to $H^s_{\alpha+1}$. In this case, by (2.10) and Corollary 3.4, if

$\rho(\overline{H}^s_{\alpha+1})=\mathrm{Min}\{(N(\alpha)-s)r,\binom{\alpha+r+1}{r-1}\}$, then $\{P_1,...,P_s\}$ satisfy the I.G.C..

4. IMPLEMENTATION IN THE CASES $k=\mathbb{Q}$ $k=\mathbb{Z}p$. REDUCTION MODULO p.

The previous algorithms have been implemented by the author in the case $k=\mathbb{Q}$ and $k=\mathbb{Z}p$. Note that the integers $H(A,n)$ and $\nu(I)$ are preserved under field extension, then an example over these fields gives an example over any field.

If $k=\mathbb{Q}$ the previous algorithm has been implemented on Macsyma on a Vax Station 2000 (with VMS Operating system). Since the elements of the matrices G^s_d are product of powers, the numbers involved in these computations can get very large so the common languages are not suitable.

On the base field $k=\mathbb{Z}p$, an implementation of the previous algorithm haas been written on a compiled version of Basic for IBM PC, but pratically any language and any computer works (clearly p should not exceed the largest integer that can be represented exactly on the computer).

In some noteworthy cases it is possible to reduce problems from $k=\mathbb{Q}$ to $k=\mathbb{Z}p$.

If $M=(m_{ij})\in M_{m,n}(\mathbb{Z})$, we denote with $M_p=(\overline{m}_{ij})\in M_{m,n}(\mathbb{Z}_p)$ the matrix whose elements are the classes of m_{ij} in $\mathbb{Z}p$.

Lemma 4.1 The following conditions hold:
(a) $\rho(Mp)\leq\rho(M)$;
(b) $\rho(Mp)=\rho(M)$ except a finite number of primes p.

Proof. It is a simple exercise.

Definition 4.2 If $M\in M_{m,n}(\mathbb{Z})$, the <u>null space</u> $N\subset\mathbb{Q}^n$ of M is the null space of M as element of $M_{m,n}(\mathbb{Q})$.
If $v=(a_1,...,a_n)\in(\mathbb{Z}^n)$, we set $\overline{v}=(\overline{a}_1,...,\overline{a}_n)\in\mathbb{Z}p$.

Lemma 4.3 Let $M\in M_{m,n}(\mathbb{Z})$. If $\rho(M)=\rho(M_p)$ (p prime) and $v_1,...,v_q\in\mathbb{Z}^n$ are elements of N then $\{v_1,...,v_q\}$ is a basis of the null space N_p of M_p.

Proof. We have (*) $\dim N_p=n-\rho(M_p)=n-\rho(M)=\dim N=q$.
$\Rightarrow$) Clearly $\overline{v}_1,...,\overline{v}_q$ span V_p. Then, by (*), we have the claim.
$\Leftarrow$) By (*), it is enough to prove that $v_1,...,v_q$ are linearly independent. Thus we consider a linear combination $\lambda_1v_1+...+\lambda_qv_q=0$.

Clearly, if the λ_i are not all null, we can assume that $\lambda_i \in \mathbb{Z}$ and that p is not a common divisor of all λ_i, $i=1,\dots,q$. We have $\overline{\lambda}_i\overline{v}_i+\dots+\overline{\lambda}_q\overline{v}_q=\overline{0}$, then by assumption $\overline{\lambda}_i=0$ for any i and the claim.

Now if S is the homogeneous coordinate ring of s points $\{P_1,\dots,P_s\}$ of $\mathbb{P}^r_k$, $k=Q$, their coordinates can be chosen in the ring of integers $\mathbb{Z}$. Thus all the computations of the algorithms of section 2 and 3 can be done over $\mathbb{Z}$. Furthermore we set $\overline{P}_i, i=1,\dots,s$ for the points whose coordinates are classes in $\mathbb{Z}_p$ of integer coordinates of $\overline{P}_i$.

We denote with I_p and S_p respectively the ideal and homogeneous coordinate ring of $\{\overline{P}_1,\dots,\overline{P}_s\}$.

Corollary. 4.4 a) For any prime p and any d we have:

$H(S_p,d) \le H(S,d)$, $\nu_d(I_p) \ge \nu_d(I)$, $\nu(Ip) \ge \nu(I)$.

Equalities hold for all but a finite number of p.

b) If $\{\overline{P}_1,\dots,\overline{P}_s\}$ are in generic position (respectively satisfy the I.G.C.) in $\mathbb{P}^r_k$, $k=\mathbb{Z}_p$ for some prime p, then $\{\overline{P}_1,\dots,\overline{P}_s\}$ are in generic position (satisfy the I.G.C.) in $\mathbb{P}^r_k$, $k=\mathbb{Q}$.

Proof. a) Follows from (1.6), (2.5), (2.7), Lemma 4.1 and Corollary 4.3 b) By definition 3.1 and (3.5), fixed s and r, if $\{\overline{P}_1,\dots,\overline{P}_s\}$ are in generic position (respectively satisfy the I.G.C.), the integers $H(S_p,d)$ $\nu_d(I_p)$ are the largest (least) possible, therefore, by a), $H(S_p,d)=H(S,d)$ and $\nu_d(I_p)=\nu_d(I)$.

Remarks 4.5. i) Using the programs written by the author in $\mathbb{Z}_p$, one checks (with small computational amount of time) tha, fixed s, the following group of s points of $\mathbb{P}^r_k$, $P_n=(2^n,3^n,5^n,7^n,11^n,\dots,p^n\dots)$, (r-times, p prime), $i \le n \le s$, satisfy the I.G.C.for a limited number of points. For example the points $(2^n,3^n,5^n,7^n)\in \mathbb{P}^3_k$, $s \le 56$, and $(2^n,3^n,5^n,7^n,11^n)\in \mathbb{P}^4_k$, $s \le 70$, satisfy the I.G.C. Then we conjecture that points P_n satisfy the I.G.C. for any s and r (if this is true then clearly the I.G.C. is proved). ii) Part b) of Corollary 4.4 was proved in **[R]**, Theorem 3.5.

REFERENCES.

[B] E. Ballico, Generators for the homogeneous ideal of s general points in $\mathbb{P}^3$, J. Algebra, 106, (1987), 46-52.

[DGM] E. Davis, A. Geramita and P. Maroscia, Perfect homogeneous ideals: Dubreil's theorems revisited, Bull. Sc. Math. 2° serie, 108, (1984), 143-185.

[H] A. Hirsthowitz, Letter to R.Hartshorne (1983).
[GM] A. Geramita and P. Maroscia, The ideal forms vanishing at a finite set points in $\mathbb{P}^n$, J. Algebra, (90), (1984), 528--555.
[GO_1] A. Geramita and F. Orecchia, On the Cohen-Macaulay type of s-lines in $\mathbb{A}^{n+1}$, J. Algebra, 70 (1981), 116-140.
[GO_2] A. Geramita and F. Orecchia, Minimally generating ideals defining certain tangent cones, J. Algebra, 78, (1982), 36-57.
[O] F. Orecchia, Points in generic position and conductors of curves with ordinary singularities, J. London Math. Soc.,24, (1981), 85-96.
[R] L. Roberts, The ideal generation conjecture for 28 points in $\mathbb{P}^3$, Can. J. Math., 38, (1986), 1228-1238.

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LOWER PERMIAN ALBAILLELLACEA (RADIOLARIA) FROM SICILY AND THEIR STRATIGRAPHIC AND PALEOGEOGRAPHIC SIGNIFICANCE

Nota di Raimondo CATALANO (*), Pietro DI STEFANO (*) & Heinz KOZUR (**)

Presentata dal Socio ordinario Bruno D'ARGENIO
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ABSTRACT

For the first time Lower Permian Radiolaria are described from Western Sicily, so far unknown from the whole Tethyan Eurasia. The fauna derives from olistoliths of dark gray, partly silty, hard marlstones and marly limestones without any macrofauna and from radiolarian micrites. Radiolarians, clearly dominated by Albaillellacea DEFLANDRE, 1952 (*Spinodeflandrella? siciliensis* n. sp., *Pseudoalbaillella scalprata scalprata* HOLDSWORTH & JONES, 1980, *P. scalprata postscalprata* ISHIGA, 1983, *P. scalprata praescalprata* n. subsp., *P. (Kitoconus) elongata* ISHIGA & IMOTO, 1980), are mostly the only fossils in these olistoliths. These radiolarian faunas were until now known only from the Circumpacific area (*P. scalprata scalprata* also known from West Texas). They give the first evidence for the presence of pelagic Lower Permian in Sicily (topmost part of the *Parafollicucullus lomentarius* Assemblage-Zone of highest Artinskian or Lower Kungurian=Lower Jachtashian, *P. ornatus* Zone of Kungurian age and perhaps *Pseudoalbaillella rhombothoracata* A.-Z. of Late Kungurian to Early Chihsian age).

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From a paleogeographic point of view the findings of these faunas suggest that a broad pelagic belt connected the Circumpacific area and Sicily during the Lower Permian. This belt, as long as the Alpidic Orogen, was situated north of Gondwana and south of the later Tethyan median platforms.

After the discovery of Middle Permian and Upper Permian (Abadehian, Dzhulfian) Albaillellaria, pelagic conodonts and psychrospheric Ostracoda, described in separate papers, can be assumed that the studied zone (Sicanian paleogeographic domain) was during the whole Permian an epioceanic area. Pelagic conditions lasted in this belt during the Mesozoic and Early Cenozoic. Apulia in the Permian was separated from Gondwanaland, but during its Mesozoic evolution was more near related to Africa.

RIASSUNTO

Viene descritta un'associazione a radiolari del Permiano inferiore proveniente da depositi pelagici della Sicilia Occidentale. Questa associazione, rinvenuta finora nell'area Circumpacifica, non era mai stata segnalata in regioni della Tetide Euroasiatica.

La microfauna a radiolari proviene dalla disgregazione di marne siltose grigio nerastre, calcari grigio scuri contenenti microbrecce e calcilutiti a radiolari. I campioni esaminati fanno parte di olistoliti di varia dimensione ed età immersi in una matrice di argille grigie piritizzate datate al Kubergandiniano (Permiano medio). Esse rappresentano uno dei termini più antichi della successione permo-triassica del Torrente San Calogero affiorante nella Valle del Sosio (Sicilia Occidentale) e recentemente descritta dagli Autori della presente nota (CATALANO *et al.*, 1988 a e in stampa).

Con l'eccezione di pochissime Cenosphaera mal conservate, tutti i radiolari appartengono alla superfamiglia Albaillellacea DEFLANDRE, 1952 (*Spinodeflandrella*? *siciliensis* n. sp., *Pseudoalbaillella scalprata scalprata* HOLDSWORTH & JONES, 1980, *P. scalprata postscalprata* ISHIGA, 1983, *P. scalprata praescalprata* n.

subsp., *P. (Kitoconus) elongata* ISHIGA & IMOTO, 1980).

L'associazione suddetta permette di riconoscere per la prima volta l'esistenza di depositi pelagici del Permiano inferiore in Sicilia. L'intervallo rappresentato corrisponde alla parte sommitale della zona a *Parafollicucullus lomentarius* dell'Artinskiano superiore- Kunguriano inferiore (=Jachtasiano inf.), alla Zona a *P. ornatus* del Kunguriano e, dubitativamente, alla Zona a *Pseudoalbaillella rhombothoracata* del Kunguriano superiore - Chihsiano inferiore.

Dal punto di vista paleogeografico, il rinvenimento di questi radiolari circumpacifici suggerisce l'esistenza di un ampio dominio pelagico che congiungeva la Regione Circumpacifica con l'area siciliana durante il Permiano inferiore. Questo dominio (Tetide Permiana) sarebbe stato situato a nord di Gondwana e a sud delle future piattaforme intermedie Tetidee.

La recente segnalazione nella successione permo-triassica della Valle del Sosio di depositi di mare profondo a radiolari (Albaillellacea) e ostracodi psicrosferici del Permiano medio e superiore (CATALANO *et al.*, 1988 a) permette di riconoscere nell'area siciliana, corrispondente al dominio paleogeografico Sicano (CATALANO & D'ARGENIO, 1978), una persistenza di condizioni pelagiche durante il Permiano, condizioni che si sono protratte nel Mesozoico e nel Terziario inferiore.

Nel Permiano, Apulia era separata da Gondwana; ben differente è la sua evoluzione mesozoica collegata a quella dell'Africa stabile.

1 - INTRODUCTION

Until now, Lower Permian Radiolaria have been described only from the Cis-Ural (KOZUR, 1980, 1981; NAZAROV & ORMISTON, 1983, 1984, 1985; NAZAROV & RUDENKO, 1981), from West Texas (CORNELL & SIMPSON, 1985, 1986) and above all from the Circumpacific area (Japan: ISHIGA & IMOTO, 1980; ISHIGA, 1982, 1983, 1985, 1986; HATTORI & YOSHIMURA, 1982; ISHIGA, KITO & IMOTO, 1982 a,b; KOJIMA, 1982; NISHIZONO, OHISHI *et al.*, 1982; SATO, NISHIZONO & MURATA, 1982; ISHIGA, IMOTO *et al.*, 1984; ISHIGA & SUZUKI, 1984; ISHIGA, WATASE & NAKA, 1986; NISHIMURA & ISHIGA, 1987; Alaska, California, Nevada, Oregon: HOLDSWORTH & JONES, 1980; HOWELL, 1986; BLOME, JONES *et al.*, 1986; Chile: LING, FORSYTHE & DOUGLASS, 1985; LING & FORSYTHE, 1987).

The distribution centre of the Permian albaillellid Radiolaria was in the Pacific and its surrounding areas. During the Early Permian time, radiolarians were also frequent in the Boreal sea with the Ural geosyncline as integral part of the Boreal faunal realm. But in the Boreal Permian the Albaillellacea were not so dominating like in the Circumpacific area. From the Pacific radiolarian domain, where Albaillellacea dominated, Permian radiolarians invaded the adjacent seas both in America and Asia (Nevada, Oregon, West Texas, South China), when pelagic conditions were present in these areas. But seemingly at the northern margin of Gondwana existed a broad pelagic sea-way, in which the Lower Permian Radiolaria (and pelagic conodonts) could migrate to the west as far as Sicily. Therefore the discovery in Western Sicily of Lower Permian Radiolaria (and pelagic conodonts, as well as Middle and Upper Permian Albaillellacea and psychrospheric ostracods, not described in this paper) is very important for Permian paleogeographic reconstructions within the Tethyan realm.

2 - LOCATION OF THE INVESTIGATED AREA AND GEOLOGICAL SETTING

The investigated radiolarian fauna comes from the section of the Torrente San Calogero, outcropping near the Pietra di Salomone (Sosio Valley), already investigated by the present authors (Figs. 1, 2).

Along this section several lithologic units, covering an Early Permian - Late Triassic time span, have been recognized (CATALANO *et al.* , 1988 a, b, and in press) that appear arranged in an overturned structure lying above a thrust plane.

The oldest rocks come from a unit consisting of badly exposed flysch deposits followed by well exposed gray pyritic soft shales with numerous olistoliths (Unit A). The matrix of these beds contains some (badly preserved) pyritized Albaillellidae (*Spinodeflandrella* sp.) and conodonts, like *Mesogondolella* cf. *phosphoriensis* (YOUNGQUIST *et al.*), *Sweetognathus subsymmetricus* WANG *et al.*, that indicate basal Middle Permian age (Kubergandinian). The olistoliths consists of: 1) large blocks of gray sandstones-siltstones with graded bedding, partly with plant detritus (flyschoid deposits); 2) dark-gray, silty, hard marls, at place with Lower Permian Albaillellacea; 3) Chishian conodont-bearing dark-gray, marly limestones containing intraformational breccias of gray, marly limestones; 4) radiolarian-bearing calcilutites with many Albaillellacea, other radiolarians and Kungurian conodonts, *e.g.* *Mesogondolella intermedia* (IGO); 5) calcarenites with *M. zsuzsannae* KOZUR (Chihsian); 6) reef-slope limestones, with *M. zsuzsannae* KOZUR, *M. slovenica* (RAMOVŠ), *Hindeodus* sp. and other conodonts (Chihsian).

The Unit B consists of red soft shales and subordinately allodapic limestones. Especially in the lower part of the

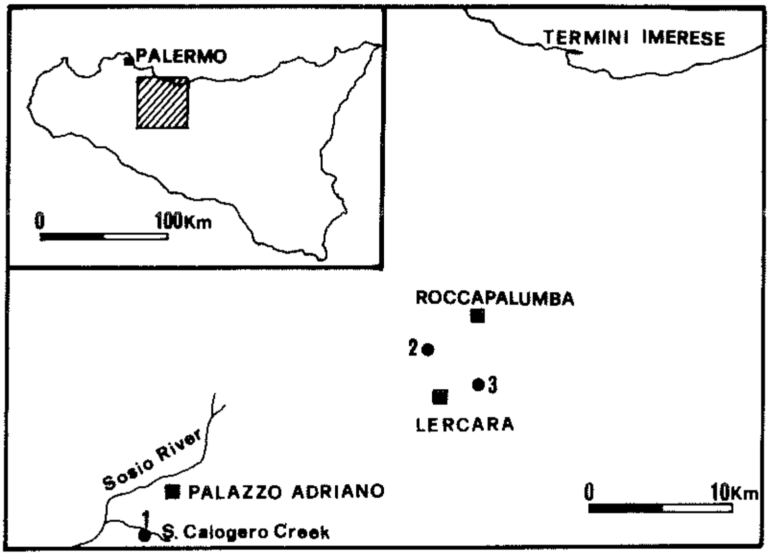

Fig. 1 - Index map of the studied areas in Western Sicily: 1) Torrente San Calogero section; 2) La Montagnola; 3) Cozzo Intronata.

unit also light-gray shales are present. The red shales have yielded very rich psychrospheric deep-water ostracod faunas of Upper Permian age and very rich and diversified Albaillellacea faunas of latest Middle Permian and earliest Upper Permian ages. In the lower part of the unit, badly preserved radiolarians and few conodonts are present. The allodapic limestones yielded no radiolarians, but ostracods and many conodonts of Middle to Upper Permian ages (Wordian - Dzhulfian). Sclerites of silicospongea are present both in the shales and in the allodapic limestones. In the latter rocks they are very common. Permian psychrospheric ostracods have been known until now only from Timor island (GRÜNDEL & KOZUR, 1975; BLESS, 1987) and Upper Permian Albaillellacea

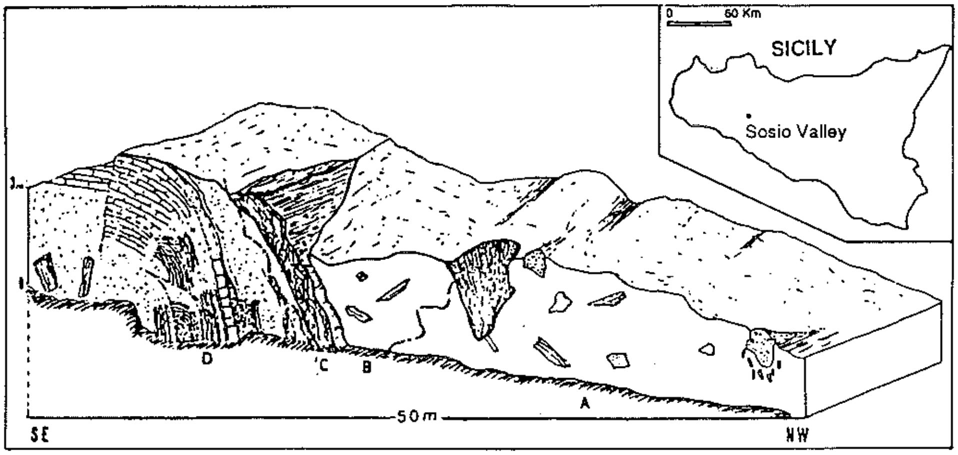

Fig. 2 - Geological sketch of the Torrente San Calogero section, WSW of Pietra di Salomone (Sosio Valley) (mod. from Catalano *et al.*, 1988 b). From NW to SE it is possible to follow an overturned sequence of tectonic slices: A) Olistostrome unit (gray shales with olistoliths), early Middle Permian. B) Red shales, Middle Permian-Upper Permian. C) Greenish siliceous marls, tuffites, siliceous to cherty gray and red radiolarites, Lower Ladinian. D) Greenish-gray and red nodular cherty limestones and marls, thin red radiolarites, Upper Ladinian to Cordevolian. The figured material was collected in the Units A and B.

have been described so far only from Japan (*see* ISHIGA *et al.*,1982 b; ISHIGA, 1986). Because of the outstanding importance of these faunas, they will be described in separate papers.

The Unit C consists of greenish-gray, gray and red radiolarites, tuffites, siliceous to cherty limestones and siliceous very hard fine-laminated radiolarian marls with well preserved Fassanian (Lower Ladinian) radiolarians.

The Unit D consists of red to greenish-gray, partly cherty, nodular limestones with intercalations of red, greenish-gray, violet marly clays and thin red radiolarites. The age is Longobardian (Upper Ladinian) to Lower Cordevolian (basal Carnian). This part of the sequence is rich in pelagic fossils, *e.g.* many ammonoids (only cross sections), bivalves (*e.g.* numerous *Daonella* spp.), highly diversified radiolarian faunas, deep-water ostracods, abundant conodonts, especially frequent *Gladigondolella tethydis*, *Paragondolella*? *trammeri*, numerous representatives of the *Pseudofurnishius sosioensis-P. huddlei-P. murcianus* line and of the *Budurovignathus truempyi-B. hungaricus-B. mungoensis* line.

3 - TAXONOMIC PART (H. Kozur)

The paleontologic material comes from seven small olistoliths of dark gray, partly silty hard marlstones to marly limestones collected in the Unit A of the Torrente San Calogero section. Four of it were barren, but three ones (samples S 3 a, c, d) have yielded a very rich, but only moderately to badly preserved microfauna that consists exclusively of Radiolaria, almost totally represented by Albaillellacea DEFLANDRE, 1952. Further Lower Permian Radiolaria (including Albaillellacea) were found at La Montagnola near Roccapalumba (Fig. 1).

The superfamily Albaillellacea was revised in detail by KOZUR & MOSTLER (1989) whose systematics is here follows.

Subclass Radiolaria MÜLLER, 1858

Order Polycystida EHRENBERG, 1838

Suborder Albaillellaria DEFLANDRE, 1952 emend. HOLDSWORTH, 1969

Superfamily Albaillellacea DEFLANDRE, 1952

Family Albaillellidae DEFLANDRE, 1952

Subfamily Spinodeflandrellinae KOZUR, 1981

Genus *Spinodeflandrella* KOZUR, 1981

Type species: *Spinodeflandrella tetraspinosa* KOZUR, 1981

***Spinodeflandrella* ? *siciliensis* n. sp.**
(figs. 4.G, 5.A-C)

Derivatio nominis: According to the occurrence in Sicily (Italy).

Holotypus: the specimen on fig. 5.A; rep.-n. CK/II-15, Dept. of Geology, Palermo.

Locus typicus: Outcrop in a small gorge of the Torrente San Calogero WSW of the "Pietra di Salomone", Sosio Valley area, Sicily.

Stratum typicum: Lower Permian olistolith of dark gray hard silty marlstone within the basal Middle Permian olistostrome unit.

Material: 8 specimens.

Diagnosis: Test laterally flattened, slender conical. Apical part above the lateral wings small, straight unsegmented. Wings not fully preserved, seemingly short. After the wings the test is strongly segmented, with generally 6 segments in this part of the test. Segments hoop-like, clearly obliquely arranged. In the constrictions one ring of large pores is present that is in uncorroded specimens closed by a layer of microgranular silica. Only adjacent to the columella some pores may be open. Columella very strong, partly visible outside. Free ends of columella not preserved.

Measurements:

Length of test: 207-233 um.
Maximum width of test: 73-80 um

Occurrence: Until now known only from the locus typicus.

Remarks: The new species is similar to *"Pseudoalbaillella" annulata* ISHIGA, 1984, that is still more slender and rather subcylindrical.

Spinodeflandrella n. sp. (*"Albaillella"* sp. B sensu ISHIGA & IMOTO, 1980, non *" Albaillella" asymmetrica* ISHIGA & IMOTO, 1982; *see* KOZUR & MOSTLER, 1989) is, in turn, distally somewhat broader and the last or the last two segments have an open pore ring.

"Albaillella" sp. SATO; NISHIZONO & MURATA, 1982 is similar in outline and segmentation, but seemingly no lateral wings are present.

By the long, slender, quite unperforated test (covered pores!) *Spinodeflandrella siciliensis* n. sp., *"Pseudoalbaillella" annulata* ISHIGA, 1984 and *"Albaillella"* sp. sensu SATO; NISHIZONO & MURATA, 1982 are distinguished from *Spinodeflandrella* KOZUR, 1981. May be that they represent a new genus within the Spinodeflandrellinae KOZUR, 1981.

Pseudoalbaillella HOLDSWORTH & JONES, 1980 emend. KOZUR & MOSTLER, 1989 and *Parafollicucullus* HOLDSWORTH & JONES, 1980 emend. KOZUR & MOSTLER, 1989 are distinguished from this species group by a distinct subdivision of the test into apical cone, pseudothorax and pseudoabdomen. Moreover, in *Pseudoalbaillella* the pseudoabdomen (homolog to the part below the lateral wings) is unsegmented, whereas in *Parafollicucullus* a few large segments are present.

Family Follicucullidae ORMISTON & BABCOCK, 1979 emend. KOZUR, 1981.

Subfamily Follicucullidae ORMISTON & BABCOCK, 1979 emend.

KOZUR, 1981.

Genus *Pseudoalbaillella* HOLDSWORTH & JONES, 1980 emend. KOZUR & MOSTLER, 1989.

Type species: *Pseudoalbaillella scalprata* HOLDSWORTH & JONES, 1980.

Pseudoalbaillella scalprata scalprata HOLDSWORTH & JONES, 1980.

(figs. 3.A-G, 5.E-G)

1980 *Pseudoalbaillella scalprata* species nova - HOLDSWORTH & JONES, p. 285, fig. 1 A, B.

1983 *Pseudoalbaillella scalprata scalprata* ISHIGA, p. 2 - 3, pl. 1, figs. 1 - 18.

Material: Several 100 specimens.

Description: See at ISHIGA (1983, p. 2-3).

Occurrence: Nevada, W-Texas, Japan, Sicily.

Pseudoalbaillella scalprata postscalprata ISHIGA, 1983.

(figs. 4.C-F, 5.I and L, 6.A)

1983 *Pseudoalbaillella scalprata* m. *postscalprata* (nov.) - ISHIGA, p. 3, pl. 2, figs. 1-16.

Material: 83 specimens.

Description: See at ISHIGA (1983, p. 3).

Occurrence: Japan, Sicily.

Remarks: *Pseudoalbaillella scalprata postscalprata* was primarily regarded as a new morphotype by ISHIGA (1983). KOZUR & MOSTLER (1989) regarded it as a subspecies of *P. scalprata* HOLDSWORTH & JONES, 1980 and we follow this use (holotype: ISHIGA, 1983, pl. 2, fig. 1).

Pseudoalbaillella scalprata praescalprata n. subsp.
(figs. 3.H and I, 4.A and B.)

Derivatio nominis: Forerunner of *P. scalprata scalprata* HOLDSWORTH & JONES, 1980.

1980 *Pseudoalbaillella* sp. cf. *Ps. scalprata* HOLDSWORTH & JONES - ISHIGA & IMOTO, p. 238, pl. 2, figs. 4-8.

Holotypus: The specimen on pl. 2 fig. 4 in ISHIGA & IMOTO (1980). rep. - no. KUE PR 2 - 1, dept. of Earth Science, Kyoto University of Education.

Locus typicus: Sasayama area, 4 km north of Sasayama-cho, Taki-gun, Hyogo prefecture (Tamba district, SW Japan, *see* ISHIGA & IMOTO, 1980, p. 234, fig. 2).

Stratum typicum: Chert lenses (olistoliths ?) of Lower Permian age.

Material: 11 specimens.

Diagnosis: Apical cone relatively short, a little curved, unsegmented. Subtriangular pseudothorax large, inflated, higher than apical cone. Pseudoabdomen very short, a little curved in the same direction as the apical cone. Lateral wings begin at the lower corners of the pseudothorax. They are downward and outward directed. Along the ventral and dorsal sides of the pseudothorax a strong rib is developed in prolongation of the lateral wings that reaches upward until the lower part of the apical cone. Free ends of columella short, without lateral spines.

Measurements:

length of test: 225 - 330 um
length of apical cone: 75 - 92 um
length of pseudothorax: 125 - 150 um
length of pseudoabdomen: 25 - 42 um

Occurrence: Lower Permian of Japan and Sicily.

Remarks: Because of its better preservation the holotype is choisen from the material figured by ISHIGA & IMOTO

(1980). In our material the lateral wings (with exception of its basal parts) and the free ends of the columella are broken away. These more fragile parts of the test are preserved in a part of the Japanese material.

Pseudoalbaillella scalprata praescalprata n. subsp. is clearly the forerunner of *P. scalprata scalprata* HOLDSWORTH & JONES, 1980. In Japan, the new subspecies occurs alone in the lower *Parafollicucullus lomentarius* A.- Z., whereas after an interval without any representatives of the *P. scalprata* line in the uppermost *P. lomentarius* A. - Z. (= *P. ornatus* zone sensu KOZUR & MOSTLER, 1989) both *P. scalprata scalprata* HOLDSWORTH & JONES, 1980 and *P. scalprata postscalprata* ISHIGA, 1984 are frequent and *P. scalprata praescalprata* n. subsp. is not more present. In our material *P. scalprata scalprata* is clearly dominating, but few specimens of *P. scalprata praescalprata* n. subsp. are still present and also the first few specimens of *P. scalprata postscalprata* appears. Transitional forms between *P. scalprata scalprata* and both *P. scalprata praescalprata* and *P. scalprata postscalprata* are present. All specimens, where the apical horn is larger than or as large as the moderatly inflated pseudothorax, will be placed into *P. scalprata scalprata* HOLDSWORTH & JONES, 1980, whereas all specimens, where the apical horn is shorter than the strongly inflated pseudothorax will be placed into *P. scalprata praescalprata* n. subsp. Moreover, the pseudoabdomen is in general still shorter in *P. scalprata praescalprata* than in *P. scalprata scalprata*. These phylomorphogenetic trends continue within the *P. scalprata* line. In *P. scalprata postscalprata* the apical horn is relatively to the pseudothorax still longer and the pseudoabdomen is considerably longer than in *P. scalprata scalprata* (pseudoabdomen about as long as pseudothorax in *P. scalprata postscalprata*). Finally, in *P. rhombothoracata* ISHIGA & IMOTO, 1980 the pseudoabdomen is clearly longer than the pseudothorax. These phylomorphogenetic changes within the *P. scalprata* line yield therefore very important stratigraphic data.

Subgenus *Kitoconus* KOZUR & MOSTLER, 1989

Type species: *Pseudoalbaillella (Kitoconus) elongata* ISHIGA & IMOTO, 1981.

Pseudoalbaillella (Kitoconus) elongata ISHIGA & IMOTO, 1980.

(figs. 4.H-J, 5.D,H,J,K)

1980 *Pseudoalbaillella elongata* n. sp. - ISHIGA & IMOTO, p. 339-340, pl. 4, figs. 1-4.

Material: More than 100 specimens.

Description: See at ISHIGA & IMOTO (1980, p. 339 - 340).

Occurrence: Lower Permian of Japan and Sicily.

Remarks: In some specimens an indistinct undulation, but no real segmentation can be observed on the long pseudoabdomen.

4 - STRATIGRAPHICAL EVALUATION

The radiolarian-bearing olistoliths consist mostly of dark gray, partly silty, hard marlstones to marly limestones without macrofossils. Four olistoliths of this type were quite barren, three ones yielded rich microfaunas that consist exclusively of Radiolaria. With exception of very few badly preserved *Cenosphaera?* sp. all radiolarians belong to the Albaillellacea DEFLANDRE, 1952. The species composition is nearly identical with radiolarian faunas from Japan and partly also from other Circumpacific occurrences.

An exact correlation of our radiolarian fauna with the Japanese radiolarian zonation proposed by ISHIGA; KITO & IMOTO (1982) and ISHIGA (1986) is difficult, because our fauna contains (even in one sample) species that are reported from different zones of the Japanese radiolarian zonation. *P. scalprata praescalprata* is known there only from the lower *Parafollicucullus lomentarius* A.- Z. Joint occurrences of *P. scalprata scalprata* and *P. scalprata postscalprata* without *P. rhombotoracata* indicate the uppermost *P. lomentarius* A.-Z. of the Japanese radiolarian zonation (=*P. ornatus* Zone in the radiolarian zonation by KOZUR & MOSTLER, 1989). *P.* (*Kitoconus*) *elongata* is known in the Japanese sections only from a short interval in the upper part of the overlying *P. rhombothoracata* A. Z.

These discrepancies can be explained by our uncomplete knowledge about the true ranges of the Permian radiolarians or by reworking within the radiolarian bearing samples. *P.* (*Kitoconus*) *elongata* ISHIGA & IMOTO, 1980 has evolved from *P.* (*Kitoconus*) *elegans* ISHIGA & IMOTO, 1980 that is known from considerable older Lower Permian beds. Between the occurrences of both species a longer interval without any representatives of the subgenus *Kitoconus* can be observed in Japan. Therefore the first appearence of *P.* (*Kitoconus*) *elongata* may be considerable earlier than the hitherto known range of this species.

In the upper *P. lomentarius* A.Z. in Japan exist an interval, where until now no representatives of the *P. scalprata* line are known. Below this interval, in the lower *P. lomentarius* A.-Z., only *P. scalprata praescalprata* is known. Above this interval *P. scalprata praescalprata* is not more present and it is replaced by *P. scalprata scalprata* and *P. scalprata postscalprata*. Both subspecies are frequent here, whereas *P. rhombothoracata* is not yet present. Our radiolarian fauna, dominated by *P. scalprata scalprata*, but still with last few representatives of *P. scalprata praescalprata* and already with first specimens of *P. scalprata postscalprata*, fits well into this interval, where no representatives of the *P. scalprata* line are known until now in the Japanese radiolarian successions (upper *P. lomentarius* A.-Z.).

KOZUR & MOSTLER (1989) have revised the radiolarian zonation proposed by ISHIGA; KITO & IMOTO (1982). For the topmost *P. lomentarius* A.-Z. they introduced the *P. ornatus* zone between the *P. lomentarius* A.-Z. s.str. and the *P. rhombothoracata* A.-Z. According to KOZUR & MOSTLER (1989) the upper *P. lomentarius* A.-Z. s.str. is characterized by the joint occurrence of *P. scalprata scalprata* and *P. scalprata praescalprata* (still named as *Pseudoalbaillella* sp. cf. *Ps. scalprata*, like by ISHIGA & IMOTO, 1980). In the topmost *P. lomentarius* A.-Z. s. str. the first representatives of *P. scalprata postscalprata* appears. *P.* (*Kitoconus*) *elongata* is according to KOZUR & MOSTLER (1989) already present in the *P. lomentarius* A.-Z. According to this revised radiolarian zonation our fauna belongs to the topmost *P. lomentarius* A.-Z. s.str.

But most of the olistoliths consist of limestones or hard marls with internal reworking. For this reason we cannot quite exclude also mixed radiolarian faunas (in our case from the *P. rhombothoracta* A.-Z. with *P.* (*Kitoconus*) *elongata* and from the lower *P. lomentarius* A.-Z. (with *P. scalprata praescalprata*). Two arguments speak against this interpretation :

1) Just our radiolarian-bearing samples have not shown any sedimentary evidences for internal reworking (inside the olistolith material).

2) In the upper *P. rhombothoracta* A.-Z. the index species *Pseudoalbaillella rhombothoracta* is very frequent. In our material this species is quite missing. For this reason we cannot recover the *P. rhombothoracta* A.-Z. in our material.

According to KOZUR & MOSTLER (1989) the *P. lomentarius* A.-Z. belongs either to the topmost Artinskian or to the Lower Kungurian (Lower Jachtashian). In the very basal part of the *P. rhombothoracata* A.-Z., conodonts of Upper Jachtashian age, figured as *Neogondolella* sp. and *Sweetognathus whitei* (RHODES) by ISHIGA & IMOTO (1980), are known from Japan. The *P. ornatus* zone and the underlying *P. lomentarius* A.-Z. must be therefore older than higher Jachtashian. On the other hand, *P. lomentarius* (ISHIGA & IMOTO, 1980) has evolved from *P. anfractus* (NAZAROV & RUDENKO, 1981) that is known from the Lower Artinskian and with near related forms also in the Upper Artinskian of the Cis-Ural. Therefore the *P. lomentarius* A.-Z. is surely younger than Lower Artinskian and probably even younger than most of the Upper Artinskian (*P.* cf. *anfractus* was not figured by NAZAROV & RUDENKO, 1981; may be that this form is near related to *P. lomentarius*).

After finishing this paper further two olistoliths with Albaillellaria were discovered at La Montagnola near Roccapalumba (sample 658) and near Cozzo Intronata at Lercara (sample 577, *see* Fig. 1).

The former consists of gray, somewhat marly micritic limestones with a very rich pyritized radiolarian fauna. Besides of Albaillellaria also other Radiolaria (Entactinaria) are frequent, but badly preserved. The Albaillellaria, moderately to badly preserved, consist of *Pseudoalbaillella scalprata scalprata* HOLDSWORTH & JONES, *P. scalprata postscalprata* ISHIGA, and *P.* (*Kitoconus*) *elongata* ISHIGA & IMOTO. This fauna is seemingly only a little younger than the above discussed one. This is insofar very interesting, because also conodonts are present in this sample: *Mesogondolella intermedia* (IGO), *see* fig.6 B and C. This fauna indicates a Kungurian (Jachtashian) age, a good confirmation of the above mentioned dating of the radiolarian fauna by KOZUR & MOSTLER (1989).

In the latter (an ammomoid- and brachiopod-bearing olistolith of a limestone with intraformational breccia) few Albaillellacea, mostly *P. scalprata scalprata*, were found. The combined evaluation of the present conodonts *M. idahoensis* (YOUNGQUIST; HAWLEY & MILLER), *Sweetognathus guizhouensis* BANDO *et al.* and radiolarians allows a placement of this sample into the latest Early Permian (Chihsian).

5 - PALEOGEOGRAPHIC IMPLICATIONS

The investigated radiolarian fauna consists exclusively of species from the Circumpacific realm. Even the new species, *Spinodeflandrella ? siciliensis* n. sp. belongs to a phyl[illegible] orphogenetic line that has only representatives in this [illegible]ea and the holotype of *Pseudoalbaillella scalprata prae*[illegible]*alprata* n. subsp. was choisen from Japanese material. Beca[illegible] radiolarians are pelagic animals, a broad pelagic connection must be present over the huge distance from the Circumpacific area to Sicily during the Early Permian time.

Further on the findings in the studied outcrop of Middle Permian (Capitanian) and Upper Permian (Abadehian) deposits with Albaillellacea and other radiolarians (fig. 5.D-K) as well as psychrospheric deep-water ostracods, described in separate papers, indicate that we had during the Permian continuous deep water conditions. These deposits consist mainly of gray shales, siltstones, sandstones and marls of Lower Permian and basal Middle Permian ages and of red shales and allodapic limestones of Upper Permian age. Adjacent to these deep-water sediments we find pelagic and shallow water sediments (reef-slope and reef-limestones), known only from olistoliths in the gray shales from the basal Middle Permian and from the Middle Permian (Wordian) blocks of the Sosio Valley area.

Seemingly the deep-water conditions continued throughout the whole Mesozoic and Early Tertiary. In the Triassic, the Ladinian is represented by the radiolarite-nodular cherty limestone sequence of the Torrente San Calogero section, until now unknown from Sicily (*see* CATALANO *et al.* 1988 and in press). These sediments are overlain by pelagic Upper Triassic cherty limestones, in its lower part with some gray marly intercalations. These intercalations have been often placed into the Mufara Formation but they are facially and faunistically different from the Mufara restricted basin deposits. As already known (CATALANO & D'ARGENIO, 1978) deep water conditions continued above the Triassic until the Tertiary.

The clear faunal connections to the Circumpacific area (the rich Upper Permian Albaillellacea associations are even known only from Japan and Sicily), the presence of psychrospheric Upper Permian ostracods and the persistence of deep water sedimentation with open connection to the Permian Pacific does not argue in favour of a deep intraplatform basin. Moreover, the climate during the Kungurian and younger Permian times was so strongly arid that any shallow water connection with the Permian Pacific would have salinar conditions in its terminal parts.

Therefore we must assume epioceanic conditions in a rather broad seaway.

When looking for regional connection of the Sicilian Permian deep-water deposits, than we find shallow-water Middle Permian rocks (thick clastic rocks, partly carbonates) in Tunisia and reworked Middle and Upper Permian shallow water deposits (present as pebbles) in the Upper Scythian flyschoid rocks of the Lagonegro Basin (Southern Italy). The Permian of Tunisia could represent the southern margin and the Permian of the Lagonegro Basin the northern margin of the pelagic Permian Tethys recognized in the Sicanian Basin.

Toward the east, we find in the Phyllite Unit of Crete Island likewise pelagic Lower and Middle Permian (FÖRSTER *et al.*, 1983, 1984; KRAHL *et al.*, 1983, 1986; KOZUR & KRAHL, 1987). Further toward the east, any northern pelagic sea-way ("Paleotethys") can be excluded, because in the Caucasus as well as in Transcaucasia and in Central Iran no pelagic Lower Permian sequences are known. Really uppermost Lower Permian (Chihsian) shallow-water sequences without pelagic faunas occur in Transcaucasia and Central Iran, whereas the basal Lower Permian is quite missing. So, the pelagic Lower Permian connection must be situated between Gondwana and the later Tethyan median platforms (Menderes, Kirsehir, Kavir, Lut, Helmand, Chaman), if it was not situated across parts of the present day Indian Ocean. This view, expressed by KOZUR & KRAHL (1987) was confirmed by the discovery of pelagic ammonoid-bearing Middle Permian in NE Irak (VAŠÍČEK & KULLMANN, 1988) and red radiolarites of latest Middle Permian age in the Hawasina nappes of Oman (BLENDINGER, 1988; DE WEVER *et al.*, 1988)

The continuation toward the east can be found in the Southern Pamir, Central Afghanistan, SE-Asia. All these areas with pelagic Lower Permian have a marginal position to Gondwana. The Gondwanide influence is still clearly visible in the lowermost part of Lower Permian, where clastic sediments prevail and the Tethyan warm water fossil assemblages are not yet present. Only since the highest Lower Permian biogenic, partly reef limestones with rich occurrences of Tethyan warm water fossils (calcareous algae, highly diversified fusulinid faunas, corals, Tethyan brachiopods, crinoids, highly diversified conodont faunas) prevail. Seemingly a similar succession existed also in the source area of the well known Permian blocks of the Sosio Valley region (Sicily). The rich Tethyan warm water fossil associations begin here with the conodont-bearing biogenic limestones of Chihsian age or a little earlier and continue into the Middle Permian with the well known Wordian ammonoid, reef and fusulinid limestones from the Sosio Valley. The radiolarian-bearing samples (here studied), in turn, show lithologically still transitional character to the more clastic Lower Permian deposits and are lacking of macrofaunas as well as fusulinids.

It is hardly to imagine that this pelagic sea-way was an about 10.000 km long, wide deep-water gulf on continental crust. At least parts of this sea-way must have oceanic or suboceanic crust (*e.g.* Oman), separating the later Tethyan median platforms, already in this time, from Gondwana. During the Permian time also Apulia was separated from stable Africa by a long persisting zone probably with oceanic or thinned continental crust; during its Mesozoic history this sector was seemingly more related to Africa.

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REFERENCES

BLENDINGER, W. (1988) - Permian to Jurassic deep-water sediments of eastern Oman Mountains: Their significance for the evolution of the Arabian margin of the south Tethys. - Facies, 19, 1-32, Erlangen.

BLESS, M.J.M. (1987) - Lower Permian ostracodes from Timor (Indonesia). - Proc. Kon. Ned. Akad. Wetensch., set. B, 90 (1), 1-13, Amsterdam/Oxford/New York.

BLOME, CH.D.; JONES, D.L.; MURCHEY, B.L. & LINIECKY, M. (1986) - Geologic implications of radiolarian-bearing Paleozoic and Mesozoic rocks from the Blue Mountains province, eastern Oregon.- In: VALLIER, T.L. & BROOKS, H.C. (Eds.): Geology of the Blue Mountains region of Oregon, Idaho and Washington.- U.S. Geol. Surv., Prof. Paper, **1435**, 79-93, Washington.

CATALANO, R.; DI STEFANO, P.; GULLO, M. & KOZUR, H. (1988) - The stratigraphic and paleogeographic significance of the rich occurrences of *Pseudofurnishius* (Conodonta) in pelagic Late Ladinian-Early Carnian sediments in Western Sicily. - Atti 74o Congr. Soc. Geol. It., **A**, 114-118, Sorrento.

CATALANO, R.; DI STEFANO, P.; GULLO, M. & KOZUR, H. (in press) - *Pseudofurnishius* (Conodonta) in pelagic Late Ladinian-Early Carnian sediments of Western Sicily and its stratigraphic and paleogeographic significance. - Boll. Soc. Geol. It., Roma.

CATALANO, R.; DI STEFANO, P. & KOZUR, H. (1988 a) - First evidence of Lower Permian Albaillellacea (Radiolaria) in the Tethyan Eurasia. - Atti 74o Congr. Soc. Geol. It., **A**, 119-123, Sorrento.

CATALANO, R.; DI STEFANO, P. & KOZUR, H. (1988 b) - New results in the Permian and Triassic stratigraphy of Western Sicily with special reference to the section at Torrente San Calogero SW of Pietra di Salomone (Sosio Valley). - Atti 74o Congr. Soc. Geol. It., **A**, 126-135, Sorrento.

CATALANO, R; DI STEFANO, P. & KOZUR, H. (in press) - New data on the Permian and Triassic stratigraphy of Sicily. - N. Jb. Geol. Paläont., Stuttgart.

CORNELL, W.C. & SIMPSON, R.D. (1985) - New Permian albaillellid radiolarians from West Texas. - Micropaleontology, **31**(3), 271-279, New York.

CORNELL, W.C. & SIMPSON, R.D. (1986) - *Nabespecha leonardica*, n. gen., n. sp.: An unusual radiolarian from the Permian of West Texas.- Micropaleontology, **32**(3), 286-288, New York.

DEFLANDRE, G. (1952) - *Albaillella* nov. gen., Radiolaria fossile du Carbonifère inférieur, type d'un lignée aberrante éteinte.- C.R. Acad. Sci. Paris, **234**, 872-874, Paris.

DE WEVER, P.; BOURDILLON -de GRISSAC, C. & BECHENNEC, F. (1988): Permian age from radiolarites of the Hawasina nappes, Oman Mountains. - Geology, **16**, 912-914, Boulder.

FÖRSTER, O.; KAUFFMANN, G.; KOZUR, H.; KRAHL, J. & RICHTER, D. (1983) - Neue Daten zur Biostratigraphie und tektonischen Lagerung der Phyllit-Quarzit-Gruppe, der Trypali-Gruppe und dem prajurassischen Anteil der Tripolitza-Gruppe auf der Insel Kreta, Griechenland. In: Geologie der Alpen, 73. Jahrestagung der Geol. Ver., Kurzfassungen, 12, Berchtesgaden.

FÖRSTER, O.; KAUFFMANN, G.; KOZUR, H.; KRAHL, J. & RICHTER, D. (1984) - New aspects to the tectonic evolution of the External Hellenides during the Upper Paleozoic and Triassic.- In: 27. Intern. Geol. Congr., Abstracts, **3**, 194-195, Moscow.

HATTORI, I. & YOSHIMURA, M. (1982) - Lithofacies distribution and radiolarian fossils in the Nanjo area in Fukui Prefecture, Central Japan.- In: NAKASEKO, K. (Ed.): Proceedings of the First Japanese Radiolarian Symposium.- News Osaka Micropaleont., Spec. Vol., **5**, 103-116, Osaka.

HOLDSWORTH, B.K. & JONES, D.L. (1980) - Preliminary radiolarian zonation for Late Devonian trough Permian time.- Geology, **8**, 281-285, Boulder.

HOWELL, D. (1986) - Des terrains déplacès: les terranes.- Science (ed. franç. Sci. Amer.), **99**, 18-28.

ISHIGA, H. (1982) - Late Carboniferous and Early Permian radiolarians from the Tamba Belt, southwest Japan.- Earth Sci., **36**(6), 333-339, Tokyo.

ISHIGA, H. (1983) - Morphological change in the Permian Radiolaria *Pseudoalbaillella scalprata* in Japan.- Trans. Proc. Palaeont. Soc. Japan, N.S., **129**, 1-8, Tokyo.

ISHIGA, H. (1985 a) - Discovery of Permian radiolarians from the pelitic rocks in the middle zone of the Chichibu Belt in eastern part of Tokushima prefecture, Shikouku. - Journ. Geol. Soc. Japan, **91** (2), 155-156, Tokyo.

ISHIGA, H. (1985 b) - Discovery of Permian radiolarians from Katsumi and Oi Formations along south of Maizuru Belt, southwest Japan and its significance. - Earth Sci., **39** (3), 175-185, Tokyo.

ISHIGA, H. (1986) - Late Carboniferous and Permian radiolarian biostratigraphy of southwest Japan. - J. Geosci., Osaka City Univ., **29** (3), 89-100, Osaka.

ISHIGA, H. & IMOTO, N. (1980) - Some Permian radiolarians in the Tamba district, southwest Japan. - Earth Sci., **34** (6), 333-345, Tokyo.

ISHIGA, H.; IMOTO, N.; YOSHIDA, M. & TANABE, T. (1984) - Early Permian radiolarians from the Tamba Belt, southwest Japan. - Earth Sci., **38** (1), 44-52, Tokyo.

ISHIGA, H.; KITO, T. & IMOTO, N. (1982 a) - Middle Permian radiolarian assemblages in the Tamba district and adjacent area, southwest Japan. - Earth Sci., **36** (5), 272-281, Tokyo.

ISHIGA, H.; KITO, T. & IMOTO, N. (1982) - Permian radiolarian biostratigraphy. In: NAKASEKO, k. (Ed.): Proceedings of the First Japanese Radiolarian Symposium. - News Osaka Micropaleont. Spec. Vol., **5**, 17-26, Osaka.

ISHIGA, H. & SUZUKI, S. (1984) - Discovery of Permian radiolarian and conodonts from the Shimomidani Formation in the "Maizuru Belt", southwest Japan and its significance. - Earth Sci., **38** (3), 197-206, Tokyo.

ISHIGA, H.; WATASE, H. & NAKA, T. (1986) - Permian radiolarians from Nishiki Group in Sangun-chugoku Belt, southwest Japan. - Earth Sci., **40** (2), 124-136, Tokyo.

KOJIMA, S. (1982) - Some Jurassic, Triassic and Permian radiolarians from the eastern part of Takayama City, Central Japan. - In: NAKASEKO, K. (Ed.): Proceedings of the First Japanese Radiolarian Symposium. News Osaka Micropaleont., Spec. Vol., 5, 81-91, Osaka.

KOZUR, H. (1977) - Beiträge zur Stratigraphie des Perm. Teil I: Probleme der Abgrenzung und Gliederung des Perm. - Freiberger Forsch. - H., **C 319**, 79-121, Leipzig.

KOZUR, H. (1978) - Beiträge zur Stratigraphie des Perm. Teil II: Die Conodontenchronologie des Perms. - Freiberger Forsch. - H., **C 334**, 85-161, Leipzig.

KOZUR, H. (1980) - Ruzhencevispongidae, eine neue Spumellaria-Familie aus dem oberen Kungurian (Leonardian) und Sakmarian des Vorurals. - Geol. Paläont. Mitt. Innsbruck, 10 (4), 235-242, Innsbruck.

KOZUR, H. (1981) - Albaillellidae (Radiolaria) aus dem Unterperm des Vorulals. - Geol. Paläont. Mitt. Innsbruck, 10 (8), 263-274, Innsbruck.

KOZUR, H. & KRAHL, J. (1987) - Erster Nachweis von Radiolarien im tethyalen Perm Europas. - N. Jb. Geol. Paläont. Abh., 174 (3), 357-372, Stuttgart.

KOZUR, H. & MOSTLER, H. (1989) - Radiolarien und Schwammskleren aus dem Unterperm des Vorurals. - Geol. Paläont. Mitt. Innsbruck, Sonderbd., 2 (2), 146-275, Innsbruck.

KRAHL, J.; KAUFFMANN, G.; KOZUR, H.; RICHTER, D.; FÖRSTER, O. & HEINRITZI, F. (1983) - Neue Daten zur Biostratigraphie und zur tektonischen Lagerung der Phyllit-Gruppe und der Trypali-Gruppe auf der Insel Kreta (Griechenland). - Geol. Rdsch., 72 (3), 1147-1166, Stuttgart.

KRAHL, J.; KAUFFMANN, G.; KOZUR, H.; MÖLLER, I.; RICHTER, D.; FÖRSTER, O.; HEINRITZI, F. & DORNSIEPEN, U. (1986) - Neue Fossilfunde in der Phyllit Gruppe Ostkretas. - Z. dt. Geol. Ges., 137, 523-536, Hannover.

LING, H. Y. & FORSYTE, R. D. (1987) - Late Paleozoic pseudoalbaillellid radiolarians from southernmost Chile and their geological significance.- Geology, 15, 253-260, Boulder.

LING, H. Y.; FORSYTHE, R. D. & DOUGLASS, R. C. (1985) - Late Paleozoic microfaunas from southermost Chile and their relation to Gondwanaland forearc development. - Geology, 13, 357-360, Boulder.

NAZAROV, B. B. & ORMISTON, A. R. (1983) - New superfamily of Stauraxopolycystine Radiolaria from the Late Paleozoic of the Soviet Union and North America. - Sencenbergiana lethaea, 64 (2/4), 363-379, Frankfurt a. M.

NAZAROV, B. B. & ORMISTON, A. R. (1984) - Vozmoznaja sistema radioljaria paleozoja. In: Morfologija, ekologija i evoljucija radioljarij. - Materialj IV simpoziuma evropejskich radioljaristov EURORAD-IV, 64-87, Leningrad.

NAZAROV, B. B. & ORMISTON, A. R. (1985) - Radiolaria from the Late Paleozoic of the southern Urals, USSR and West Texas, U.S.A. - Micropaleontology, 31 (1), 1-54, New York.

NAZAROV, B. B. & RUDENKO, V. S. (1981) - Nekotorye bilateralnosimmetricnye radioljarii pozdnego paleozoja juznogo Urala. - Vopr. mikropaleont., **24**, 129-139, Moskva.

NISHIMURA, K. & ISHIGA, H. (1987) - Radiolarian biostratigraphy of the Maizuru Group in Yanahara area, southwest Japan. - Mem. Fac. Sci., Shimane Univ., **21**, 169-188, Matsue.

NISHIZONO, Y.; OHISHI, A. *et al.* (1982) - Radiolarian fauna from the Paleozoic and Mesozoic formations, distributed along midstream of Kuma river, Kyushu, Japan. - In: NAKASEKO, K. (Ed.): Proceedings of the First Japanese Radiolarian Symposium. - News Osaka Micropaleont., Spec. Vol., **5**, 311-326, Osaka.

ORMISTON, A. R. & BABCOCK, L. (1979) - *Follicucullus*, new radiolarian genus from the Guadalupian (Permian) Lamar Limestone of the Delaware Basin. - J. Paleont., **53** (2), 328-334, Lawrence.

SATO, T.; NISHIZONO, Y. & MURATA, M. (1982) - Paleozoic and Mesozoic radiolarian faunas from the Shakumasan Formation. - In: NAKASEKO, K. (Ed.): Proceedings of the First Japanese Radiolarian Symposium. - News Osaka Micropaleont., Spec, Vol., **5**, 301-310, Osaka.

SHENG, Jin-zhang & WANG, Yu-jing (1982) - Fossil Radiolaria from Kufeng Formation at Longtan, Nanjing. - Acta Paleont. Sinica, **24** (2), 171-180, Beijing.

VAŠÍČEK, Z. & KULLMANN, J. (1988) - Eine Ammonoiden-Fauna vom Sosio-Typ (Guadalupian, Oberperm) aus Kurdistan (Irak). - Paleontographica, A, **204** (1-3), 95-115.

Fig. 3 – A–G: *Pseudoalbaillella scalprata scalprata* HOLDSWORTH & JONES, x 210, A: rep.–no. CK/II–21, B: rep.–no. CK/II–27, C: rep.–no. CK/II–61, D: rep.–no. CK/II–14, E: rep.–no. CK/II–12, F: rep.–no.II–13, G: lateral view, rep.–no. CK/II–9. – H and I: *Pseudoalballella scalprata praescalprata* KOZUR, n. subsp., x 210, H: lateral view, rep.–no. CK/II–10, I: rep.–no. CK/II–8. All specimens are from the Torrente San Calogero section WSW of Pietra di Salomone (Sosio Valley, Western Sicily), sample S 3a, dark–gray, silty hard marl of latest Artinskian or more probably Early Kungurian (= Early Jachtashian) age, uppermost *Parafollicucullus lomentarius* A.–Z. sensu KOZUR & MOSTLER, olistolith from the basal Middle Permian (Kubergandinian) Olistostrome Unit (Unit A in fig. 2). Collections of Dipartimento di Geologia e Geodesia, Palermo University.

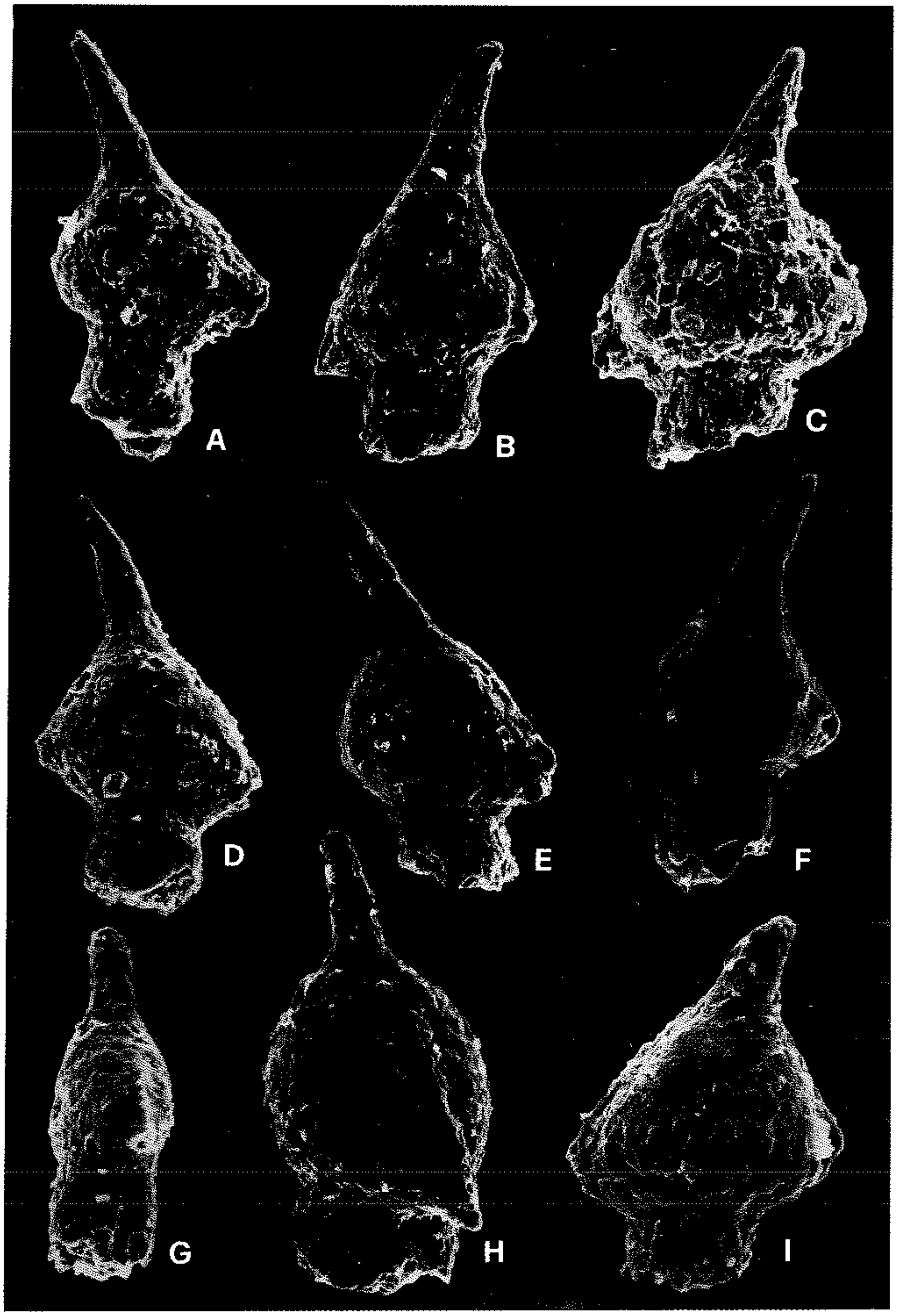
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Fig. 4 – A and B: *Pseudoalbaillella scalprata praescalprata* KOZUR, n. subsp., x 210, A: rep.–no. CK/II–7, B: lateral view, rep.–no. CK/II–11. – C–F: *Pseudoalbaillella scalprata postscalprata* ISHIGA, x 210, C: rep.–no. CK/II–4, D: rep.–no. CK/II–5, E: rep.–no. CK/II–6, F: rep.–no. CK/II–66. – G: *Spinodeflandrella ? siciliensis* KOZUR, n. sp., apical part broken away, x 310, rep.–no. CK/II–19. – H–J: *Pseudoalbaillella (Kitoconus) elongata* ISHIGA & IMOTO, x 210, H: rep.–no. CK/II–1, I: rep.–no. CK/II–2, J: rep.–no. CK/II–3. All specimens are from sample S 3 a (see fig. 3). Collections of Dipartimento di Geologia e Geodesia, Palermo University.

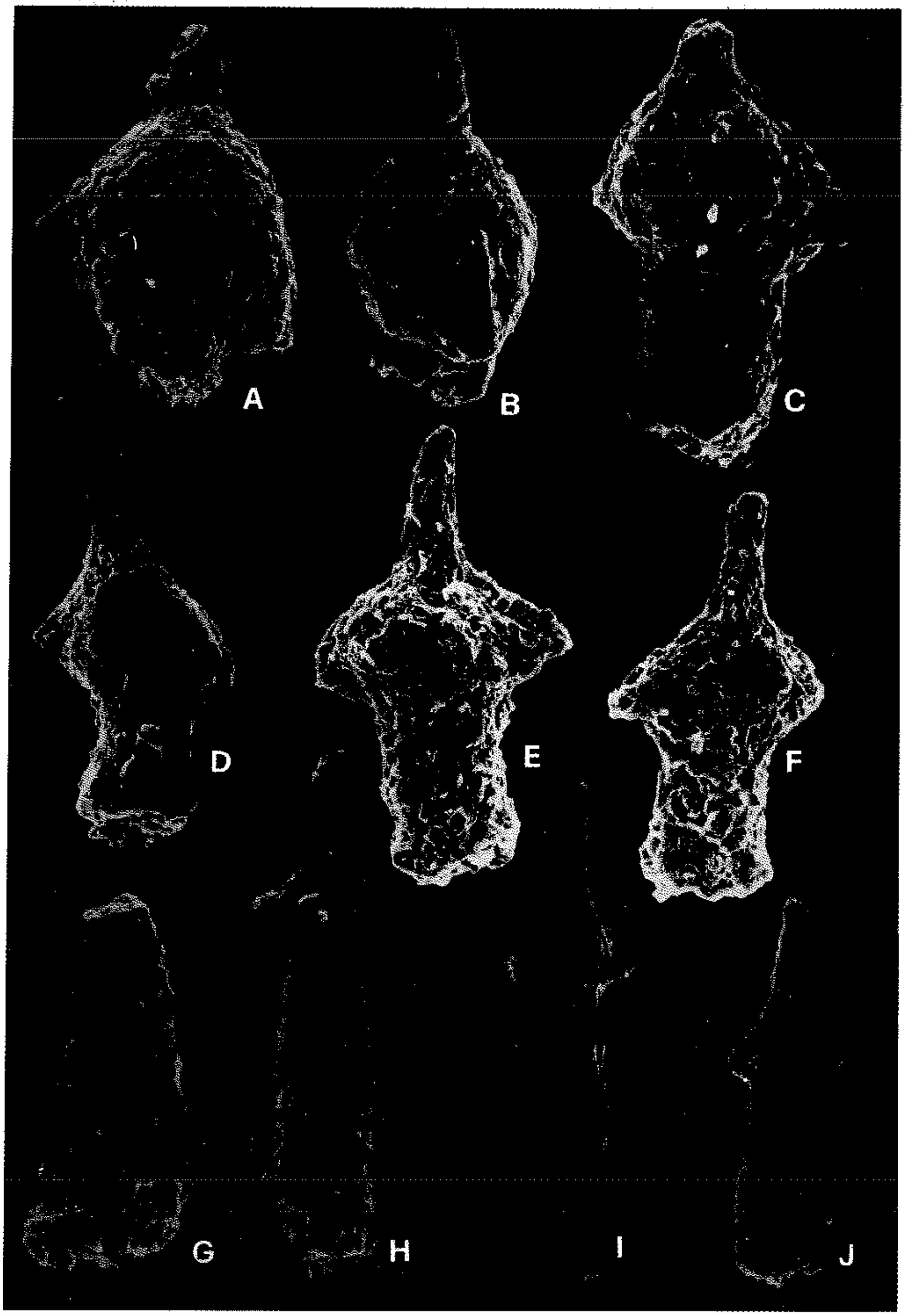
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Fig. 5 – A–C: *Spinodeflandrella ? siciliensis* KOZUR, n. sp., x 310, A: holotype, rep.–no. CK/II–15, B: rep.–no. CK/II–16, C: lateral view, rep.–no. CK/II–17. – D, H, J, K: *Pseudoalbaillella (Kitoconus) elongata* ISHIGA & IMOTO, x 170, D: rep.–no. CK/V–50, H: rep.–no. CK/V–54, J: rep.–no. CK/VI–25, K: rep.–no. CK/VI–26. – E–G: *Pseudoalbaillella scalprata scalprata* HOLDSWORTH & JONES, x 170, E: rep.–no. CK/V–53, F: rep.–no. CK/V–49, G: rep.–no. CK/VI–29.– I and L: *Pseudoalbaillella scalprata postscalprata* ISHIGA, x 170, I: rep.–no. CK/V–52, L: rep.–no. CK/VI–28. The specimens 5. A–C are from sample S 3a (see fig. 2). The specimens 5. D–L are pyritized radiolarians from sample 658, olistolith of gray radiolarian calcilutite of Kungurian (Jachtashian) age *(Parafollicucullus ornatus* Zone) from the basal Middle Permian Olistostrome Unit (Unit A in fig. 1), locality and repository place as in fig. 3.

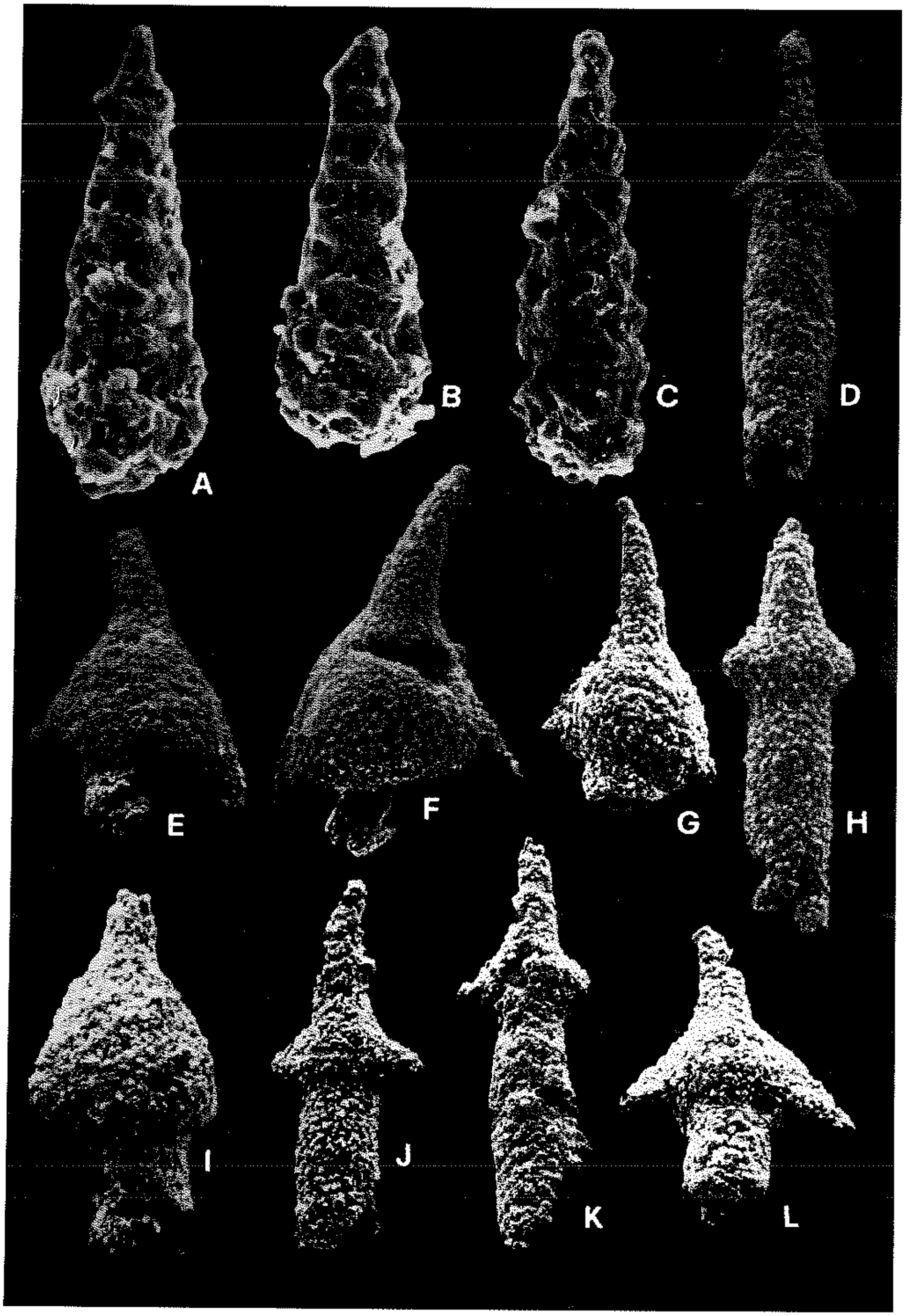
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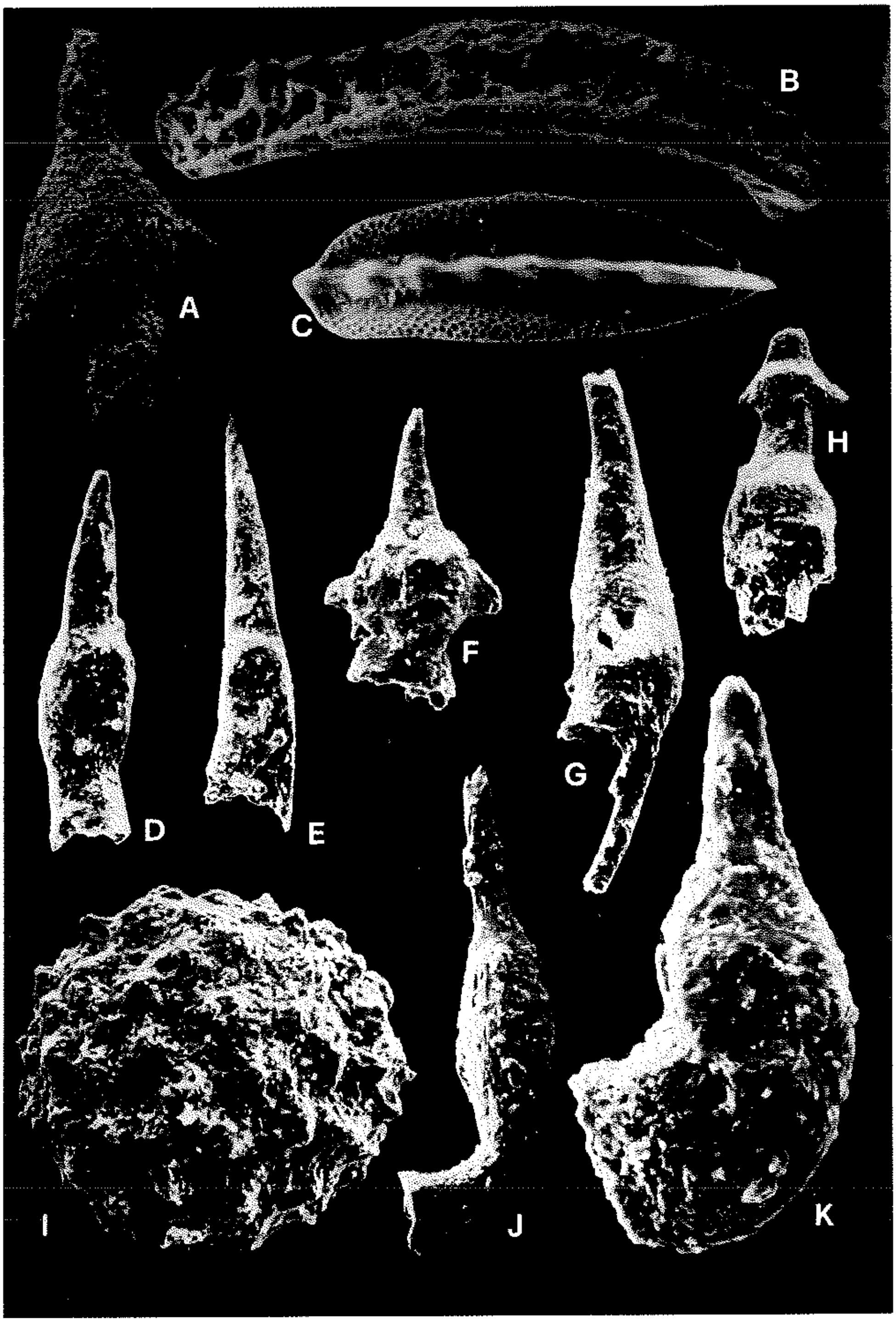
Fig. 6 – A: *Pseudoalbaillella scalprata postscalprata* ISHIGA, x 170, rep.–no. CK/V–51. – B and C: *Mesogondolella intermedia* (IGO), x 170, B: adult specimen, oblique lateral view, rep.–no. CK/V–48, C: juvenile specimen, upper view, rep.–no. CK/V–47. – D: *Follicucullus* n. sp. (=*Follicucullus scholasticus*, morphotype II sensu ISHIGA, 1985), x 160, rep.–no. CK/III–81. – E: *Ishigaconus scholasticus* (ORMISTON & BABCOCK), x 160, rep.–no. CK/III–70. – F: *Pseudoalbaillella eurasiatica* KOZUR; KRAHL & MOSTLER, x 170, rep.–no. CK/III–50. – G: *Ishigaconus* ? n. sp. A, x 170, rep.–no. CK/III–27. – H: *Parafollicucullus* n. sp. (=*Pseudoalbaillella* sp. cf. *fusiformis* sensu NISHIMURA & ISHIGA, 1987), x 170, rep.–no. CK/III–26. – I: *Phaenicosphaera mammifera* (NAZAROV & ORMISTON), x 170, rep.–no. CK/III–59. – J: Highly evolved Follicucullidae, n. gen. n. sp., x 160, rep.–no. CK/III–52. – K: *Follicucullus* ? n. sp. aff. *charveti* CARIDROIT & DE WEVER, x 320, rep.–no. CK/III–45. The specimens 6.A–C are from sample 658 (see fig. 5.D–L). The specimens 6.D–K are from sample 655, red, soft clay with psychrospheric deep–water ostracods and mass occurrences of Circumpacific radiolarians (Unit B in fig. 2), latest Middle Permian to lower part of Late Permian (*Follicucullus ventricosus – Ishigaconus scholasticus* A.–Z. to *Follicucullus ? charveti – Imotoella triangularis* A.–Z.). Locality and repository place as in fig. 3.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SULL'ESTRAZIONE DEL CALORE DALLE ROCCE CALDE SECCHE

Nota di de'Medici G.B.(°) Mazza S.(°) Sinno R.(°°)

presentata dal socio ordinario Bruno D'Argenio e dal socio corrispondente Tullio Pescatore.

Adunanza del 4 Novembre 1989

RIASSUNTO

Nel presente lavoro sono descritti i serbatoi geotermici classici di tipo idrotermale,per lasciar spazio,poi,al processo di estrazione del calore dalle rocce calde secche. Infine sono paragonati i potenziali termici sfruttabili di tali tipi di serbatoi con l'equivalente di un giacimento di idrocarburi.

ABSTRACT

In the present work the main geothermal classic reservoirs of hydrothermal type are described with reference to the hot dry rock process. A comparison between the exploitable thermal potentiality of that type of reservoir and the equivalent of a hydrocarbon layer is also reported.

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Com'e' noto,specie in questi ultimi anni di risveglio dei problemi energetici e delle cosi' dette fonti "pulite" ed alternative,quando si parla di riserve geotermiche ci si riferisce,il piu' delle volte,ai serbatoi geotermici tradizionali cioe' agli accumuli endogeni di tipo idrotermale. E' pure noto che detti giacimenti possono essere schematicamente classificati sia in funzione delle caratteristiche meccaniche e geologiche delle rocce,sia della natura chimico-fisica del fluido in esse contenuto. Le Figg.1 e 2 ci danno l'idea dei due schemi fondamentali con cui puo' essere inquadrato un campo geotermico tradizionale. [Smith M.C.,Mc Farland R.D..(1)]. Nel primo caso e' illustrato un sistema geotermico con acquifero confinato, costituito da una fonte di calore (plutone intrusivo),da una roccia serbatoio e da una copertura impermeabile. Il serbatoio e' costituito da una roccia permeabile (in genere calcarea), fratturata,in contatto con i circuiti delle acque di falda. La roccia impermeabile consente di determinare condizioni favorevoli all'accumulo del fluido nel sottosuolo,e ad impedire lo scambio termico,per convezione,con le rocce sovrastanti. In questo sistema il calore viene trasmesso verso l'alto per mezzo di

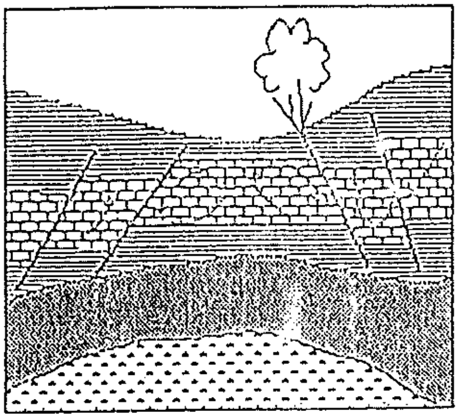

Fig.1 - Sistema geotermico con acquifero confinato.
Plutone intrusivo. Roccia serbatoio.
Roccia incassante. Roccia imp.di cop.

circuiti di convezione chiusi,che si originano all'interno dello acquifero confinato. L'acqua si trova in pressione e la circolazione di tipo forzato,tende ad omogeneizzare la temperatura del serbatoio. Nei sistemi ad acquifero libero,Fig.2, invece,non avendosi la copertura impermeabile,il fluido dei circuiti convettivi aperti trasmette il calore verso l'alto, addirittura fino alla superficie.

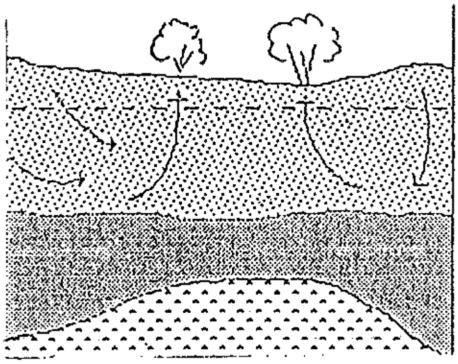

Fig.2 - Sistema geotermico con acquifero libero.
Plutone intrusivo. Roccia incassante.
Acquifero.

In presenza di un acquifero libero,con gradiente geotermico normale,non possono ottenersi concentrazioni di calore significative,anche in considerazione del fatto che l'acquifero viene a trovarsi in contatto con l'acqua di infiltrazione a bassa temperatura. A tale proposito va ricordato che si intende per gradiente geotermico la differenza di temperatura,in gradi centigradi,esistente tra due superfici distanti 100 metri una dall'altra lungo la verticale nella crosta terrestre superiore. Un gradiente geotermico medio di 3°C/100m rappresenta un valore normale. Relativamente alla natura dei fluidi,i "reservoirs" geotermici possono essere costituiti da accumuli di acqua calda pressurizzata,in condizioni ipercritiche,da serbatoi di vapore saturo e secco,o surriscaldato (caso migliore per un corretto sfruttamento industriale) ed,infine,da miscele liquido-vapore,
con la fase liquida preponderante ("water dominated") oppure con il vapore in quantita' nettamente maggiore ("dry steam").
Esaurita questa premessa,necessaria per una logica e corretta articolazione della materia in esame,c'e' da osservare che esistono numerose zone geotermiche,non necessariamente caratterizzate da

eruzioni di vapore o di acqua calda,ma che presentano,per contro,elevati ed anomali gradienti geotermici, anche a profondita' non eccessive,facilmente accessibili. Si tratta,in effetti,di formazioni rocciose che,trovandosi,per la loro collocazione geologica a temperature elevate,magari a distanza non molto elevata da una sacca magmatica,rappresentano una potenziale risorsa di energia,se cosi' si puo' dire,allo "stato solido",con la grande limitazione,pero',di una scarsa,se non addirittura nulla,circolazione di acqua. Il problema diventa, pertanto,quello di creare una permeabilita' artificiale in detti sistemi,fratturandone una notevole zona,mediante l'iniezione, dall'alto,di acqua in pressione. In genere dette formazioni rocciose asciutte si trovano ad una temperatura oscillante tra i 300 ed i 500 °C ed a 3 o 4 Km di profondita'. Secondo alcuni autori,[Makarenko F.;Konov V. (2)],le aree interessate da rocce di questo tipo occuperebbero circa un decimo della superficie terrestre.

Nel caso di idrofratturazione l'acqua che viene iniettata ha una duplice funzione. Da una parte il suo impatto,a notevole pressione,con la roccia cristallina serve a creare cavita' e fratture nella

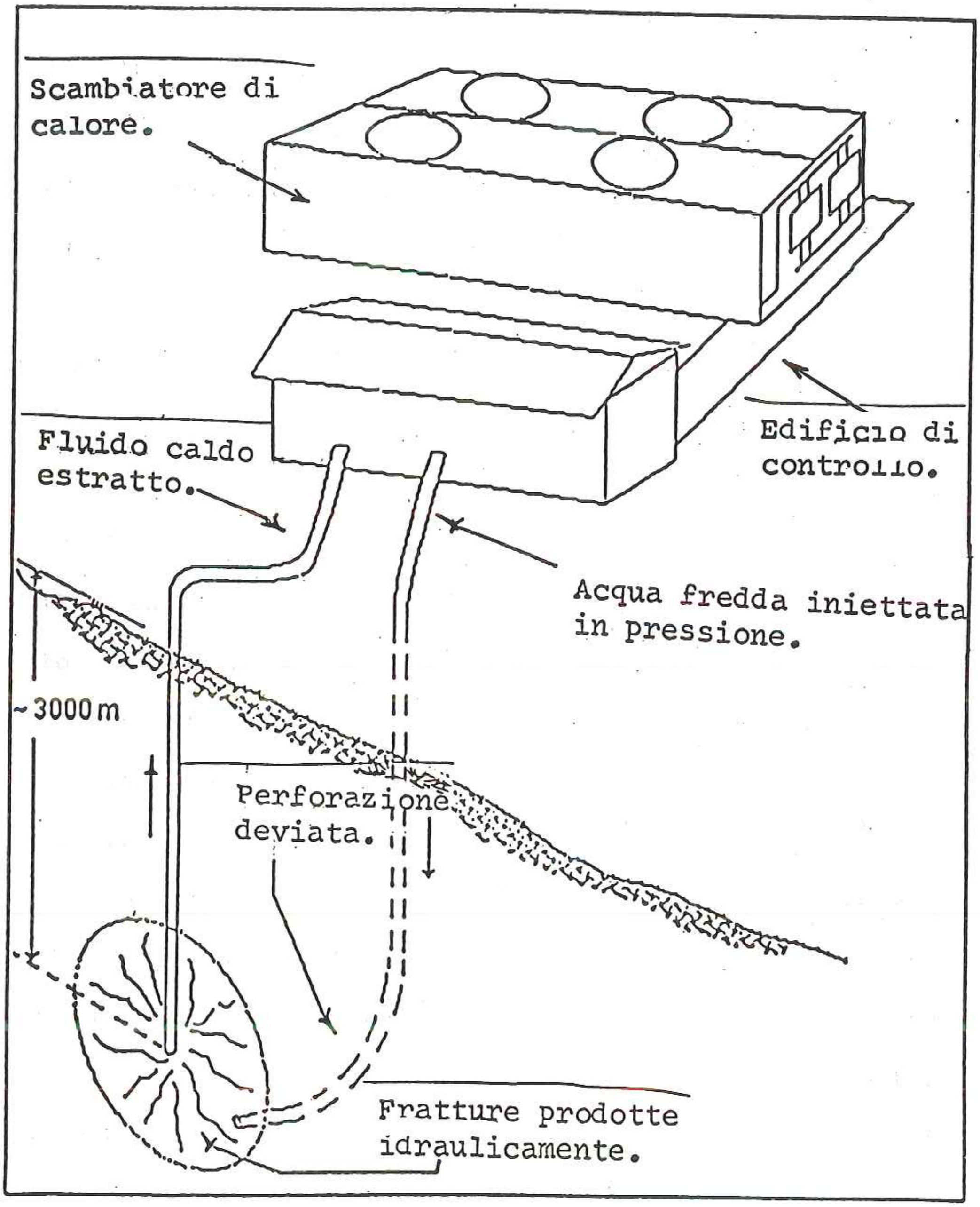

Fig. 3 Principio del procedimento " hot dry rock " per riscaldare l'acqua nella roccia plutonica surriscaldata

roccia stessa;dall'altra essa funge da fluido termovettore per l'estrazione ed il trasferimento del calore dalla roccia. Le superfici che delimitano le crepe verticali formatesi nella roccia,di pochi centimetri di spessore,fungono da scambiatori termici,mentre le cavita' sono,per cosi' dire,i "boilers" da cui viene pompata,verso l'alto,l'acqua per estrarre il calore dalla roccia. Dopo essersi riscaldata,l'acqua,in circolo sotto forma di liquido,vapore o di miscela liquido-vapore torna in superficie attraverso un secondo pozzo.Fig.3 - [Kruger P.;Ramey J.H. (3)].

La risalita avviene mediante pompaggio,che puo' essere arrestato una volta che si siano instaurati fenomeni di convezione naturale nel fluido. Il disegno mostra che la perforazione di iniezione (deviata nel tratto piu' profondo) deve incontrare la cavita' sotto la trivellazione,per l'estrazione che avviene in un secondo tempo. In superficie i due pozzi distano l'uno dall'altro tra i 20 ed i 50 metri. la massa m di una formazione di rocce calde secche,di volume V,densita' media ρ ,porosita' ϕ e calore specifico medio C e' data da :

$$m = \rho V \cdot (1 - \phi) \qquad (1).$$

Se quest'ultima subisce un abbassamento di temperatura ΔT la quantita' ideale di calore estraibile e' data da :

$$Q_i = \rho V \cdot (1-\phi) \cdot C \cdot \Delta T \qquad (2)$$

La quantita' reale di calore estraibile,invece e' data da :

$$Q_r = f \rho V \cdot (1-\phi) \cdot C \cdot \Delta T \qquad (3)$$

ove f e' un fattore correttivo,numerico,compreso nell'intervallo (0,1),che tiene conto del rendimento dell'operazione di estrazione del calore. Dalla (3) si ha :

$$\frac{Q_r}{V} = H = f \rho \left(1-\phi \right) C \, \Delta T \qquad (4)$$

in cui H,espressa in $Kcal/m^3$,rappresenta l'energia volumetrica estraibile realmente. Per una formazione rocciosa massiccia, caratterizzata dai seguenti parametri :

$\rho = 2600\ Kg/m^3$; $\phi = 0.25$; $f = 0.1$

$C = 0.30\ Kcal/Kg\ °C$; $\Delta T = 200\ °C$,

si ha dalla (4) :

$$H = (0.1) \cdot (1-0.25) \cdot (0.30) \cdot (200) \cong 12.000\ Kcal/m^3.$$

Assumendo per il petrolio un potere calorifico medio pari a 10000 $Kcal/m^3$,si ha che il suo quantitativo necessario per fornire lo stesso ammontare energetico, teste' calcolato, risulta pari a 1.2 m^3. Secondo tale risultato 1 Km^3 di roccia potrebbe fornire energia pari ad un grande giacimento petrolifero di 1200 miliardi di litri.

Studi di modellistica termofluidodinamica ed esperienze sul campo ed in laboratorio [Report LA - 4547 Los Alamos Scientific Laboratory (4)] hanno portato a due equazioni; la prima interlega le temperature T_1 e T_2 dell'acqua dell'impianto in uscita (fase di iniezione) ed all'ingresso (fase di recupero) con la temperatura media T del serbatoio geotermico e altri parametri operativi (come gradiente geotermico) : $\bar{K}$ = °C/m

$$T_2 = T_1\, e^{-Al} + \left(T - \frac{\bar{K}}{A}\right) \cdot \left(1 - e^{-Al}\right) + \bar{K}l \qquad (5)$$

ove le temperature T_i sono espresse in °C, l e' la lunghezza della sezione di calcolo in m ed A e' un parametro, caratterizzante lo scambio termico, definito da :

$$A = \frac{\Pi\, d_w\, K_r}{WC}$$

in cui dw e' il diametro del pozzo in m, W e' la produzione dei pozzi in = Kg/h.

La seconda equazione e' relativa alle zone di frattura :

$$T_3 = T_2 \cdot e^{-A_f l_f} + \left(T - \frac{K}{A_f} \right) \cdot \left(1 - e^{-A_f l_f} \right) + \bar{K} l_f \qquad (6)$$

dove T_3 e T_2 sono le temperature dell'acqua all'uscita e allo ingresso delle fratture, in °C, T e' la temperatura media, in °C, del reservoir, l_f la sommatoria delle zone lineari di frattura espresse in metri, essendo :

$$A_f = \frac{\Pi d_f K'_f}{WC}$$

in cui d_f e' il diametro della frattura, in m, con K'_f coefficiente di scambio termico nella zona di frattura espresso in Kcal/hm² °C.

Tornando al discorso di base generale c'e' da dire che la formazione di crepe nella roccia puo' essere indotta, oltre che da fratturazione idraulica, anche da processi di stress termici o da impiego di esplosivi. Dette frantumazioni servono a realizzare una permeabilita' ex-novo, oppure , ad aumentarla, esaltando il diametro di crepe preesistenti, o rimuovendo i depositi che tenderebbero a richiuderle. L'introduzione di fluidi piu' freddi in una formazione geotermica gia' fratturata, naturalmente o da metodi idraulici o da esplosivi, produrra' in essa un gradiente di temperatura, a sua volta responsabile dell'insorgere di uno stato tensionale, causante l'allargamento di crepe preesistenti o la formazione di altre nuove in seguito alla non omogenea contrazione del materiale. Come si vede,quindi,i due metodi di fratturazione idraulica e termica possono,anzi debbono, considerarsi complementari ed integrativi. La contrazione volumetrica ΔV ,dovuta ad un abbassamento di temperatura ΔT della roccia,sara' data da :

$$\Delta V = V \beta \Delta T \qquad (7)$$

dove β e' il coefficiente di espansione volumetrica,in $°C^{-1}$.

In seguito a detto abbassamento ΔT di temperatura si riesce,come gia' visto,ad estrarre dalla roccia una quantita' di calore ideale Q_i data da :

$$Q_i = \rho V (1-\phi) C \Delta T \qquad (2)$$

Dal rapporto $\Delta V / Q_i$ si ha :

$$\frac{\Delta v}{Q_i} = \frac{\beta}{\rho C (1-\phi)} \qquad (8)$$

da cui

$$\Delta v = \frac{\beta Q_i}{\rho C (1-\phi)} = \frac{Q_r \beta}{H} \qquad (9)$$

Da calcoli fatti,relativamente a masse granitiche,impiegando le equazioni ed i parametri scritti in precedenza e ricordando che per un granito e' :

$$\beta = (2.48) \cdot 10^{-5} \; [°C]^{-1}$$

si e' visto che per ottenere una produzione di energia pari a 2.4 Mw h,si verifica una contrazione globale di volume ΔV pari a $\Delta V = 100\ m^3$

Come si evince dalla (9) la quantita' di calore estratta dipende dal volume dei vuoti e,quindi,in definitiva,dalle crepe prodotte. Per impedire a queste ultime di chiudersi si aggiunge sabbia all'acqua;viceversa l'aggiunta di un gelificante all'acqua tenderebbe a stagnare nella frattura le superfici permeabili. Detti processi di estrazione del calore dalle rocce calde asciutte furono sperimentati per primi dagli americani nei laboratori "LASL" di Los Alamos nel Nuovo Messico. C'e' pero' da dire che,vuoi per la leggera stasi che ha avuto la geotermia in generale,vuoi per le difficolta' stesse,annidate in tali operazioni,queste tecniche sono ancora,dopo un certo numero di anni,rimaste ad uno stato sperimentale,quasi pionieristico. In Italia,a Cesano,nel Lazio,si e' cercato di fare,e si sta facendo, anche se con molta lentezza,qualcosa di simile.[Geotermia e Regioni - Atti Convegno Chianciano - 1977 (5)]. Si spera di poter fare altrettanto,ovviamente solo dopo attenti studi di carattere geologico e petrografico anche in Campania,dove "aree termiche secche",pure dovrebbero esservene,ed anche in proporzioni non trascurabili. In dette zone una "geotermia artificiale" potrebbe essere,addirittura,paradossalmente,da preferirsi a manifestazioni termali spontanee,dal momento che alcuni pozzi attivi "umidi",tipo il pozzo TRE-CASE 1 scavato dall'Agip nel comune di Boscotrecase (NA),hanno dovuto essere abbandonati,specialmente per l'elevato contenuto di sali e di sostanze nocive nel fluido erogato.

BIBLIOGRAFIA

1) Smith M.C.,Mc Farland R.D. : Energy from dry geothermal reservoirs,Seminar Leningrad August Soviet-American.1974.

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2) Makarenko F.,Konov V. : Geothermal Resources of the URSS.Nauka Moscow Russian.1970.

--

3) Kruger P.,Ramey J.H. : Stimulation and Reservoir Engineering of Geothermal Resource - Report 5GP-TR-1,Stanford University.1974.

--

4) A Preliminar Study of the Nuclear Subterrene,Report LA-4547, Los Alamos Scientific Laboratory.1971.

--

5) Geotermia e Regioni,Convegno Chianciano.1977.

CALCOLO DELLO STATO TENSIONALE INDOTTO SU ROCCE DA PROCESSI DI FRATTURAZIONE IDRAULICA

Nota di de'Medici G.B.(°) Mazza S.(°) Sinno R.(°°)

presentata dal socio ordinario Bruno D'Argenio e dal socio corrispondente Tullio Pescatore.

Adunanza del 4 Novembre 1989

RIASSUNTO

Il lavoro riporta la trattazione analitica degli sforzi meccanici indotti su rocce da processi di fratturazione idraulica. Infine e' confrontato il comportamento di tre tipi diversi di rocce alla luce dei risultati ottenuti.

ABSTRACT

The present work relates the mathematical description of the mechanical stress induced in rock formations by hydraulic fracturing. Finally three types of different rocks are examined for comparison on the basic of analytical results.

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Il metodo della fratturazione idraulica, com'e'noto, consiste in una tecnologia che consente di creare artificialmente, con l'immissione di acqua in pressione, condizioni di permeabilita' in masse di rocce originariamente impermeabili, ma dotate di un elevato gradiente geotermico, a profondita' non eccessive, in modo da poterne estrarre calore. [Rau H. (1)]. Detta operazione richiede a monte la conoscenza dei meccanismi di indebolimento della roccia, della elasticita', della plasticita', dell'omogeneita', dell'isotropia del materiale, degli effetti della penetrazione del fluido e cioe', sostanzialmente, dello studio dello stato di tensione e di deformazione del materiale. [Atti dei Convegni Lincei.1977. (2)]. Consideriamo un cubetto elementare di roccia posto ad una profondita' h rispetto alla superficie terrestre. Fig.1.

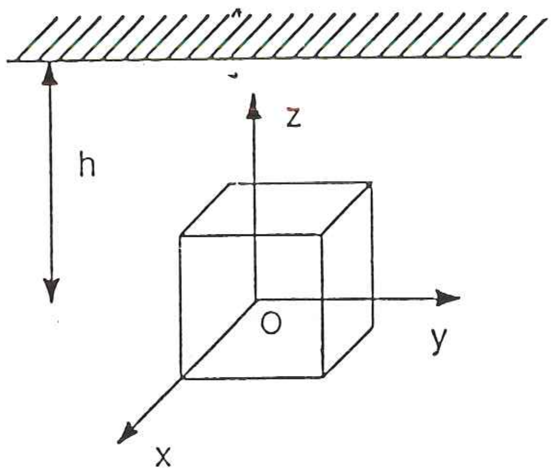

Fig. 1

Sulla faccia di normale z si esercita una pressione pari a $p_z = -(\gamma \cdot h)$ (1), dato l'orientamento scelto per l'asse z. Detta pressione sara' bilanciata dalla tensione interna(diretta lungo +z):

$$\sigma_z = (\gamma \cdot h). \qquad (2).$$

Per effetto di p_z l'elementino considerato dovrebbe allungarsi nelle direzioni x ed y, il che, ovviamente, non avviene se si considera indeformabile la restante roccia confinante che, in pratica, esercita una compressione sulle facce di normali x ed y. Per tale ragione sulle stesse facce sorge uno stato tensionale σ_x e σ_y con $\sigma_x = \sigma_y = \sigma_H$, in direzione +x e +y.

Per il calcolo di detto valore, richiamando l'osservazione di allungamenti nulli nelle direzioni x ed y, sara':

$$\frac{1}{E}\left(\sigma_y - \frac{\sigma_y + \sigma_z}{m}\right) = 0 \qquad (3),$$

essendo

$$\frac{1}{E}\left(\sigma_y - \frac{\sigma_y + \sigma_z}{m}\right) = \varepsilon_y \qquad (4)$$

Dalla (3) si ha:

$$\sigma_y = \sigma_x = \sigma_H = \frac{\sigma_z}{(m-1)} = \sigma_z \cdot \left(\frac{\nu}{1-\nu}\right) \qquad (5)$$

Nelle formule riportate e' E = modulo di elasticita' lineare della roccia o modulo di Young = Kg/cm²;

ν = 1/m = coefficiente di Poisson (numero puro).

I valori trovati per σ_z e σ_H rappresentano lo stato tensionale in tutti i punti distanti h dalla superficie terrestre, per effetto della sola pressione litostatica degli strati di roccia sovrastanti. Supponiamo ora di effettuare un foro di trivellazione di raggio r_0 nella roccia e di pomparvi un liquido (in genere acqua). Sia p_0 la sua pressione idraulica alla profondita' h. Per calcolare lo stato tensionale in un generico punto M distante r da O, dovuto alla p_0 consideriamo un sistema di coordinate polari r, θ con origine in O, centro del foro nel piano z = h (piano parallelo al piano del foglio). Fig.2.

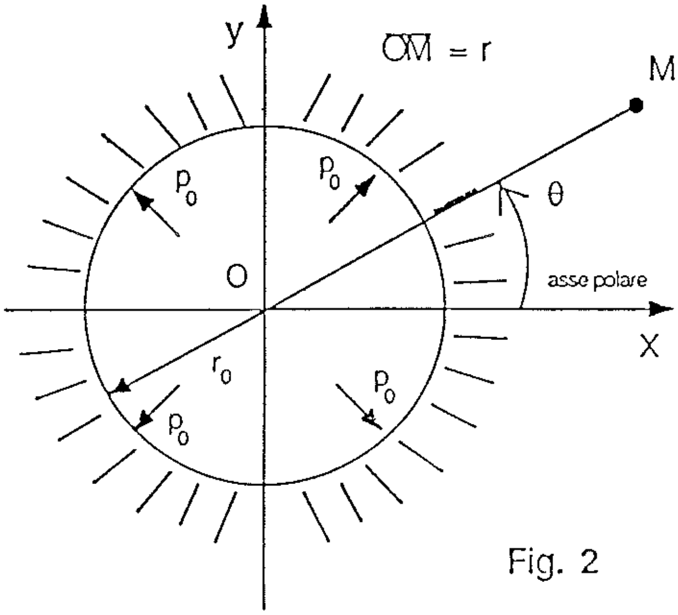

Fig. 2

Riferite a tale sistema le equazioni di equilibrio di Cauchy, trattandosi di uno stato piano di tensione, con le tensioni, cioe', indipendenti da z, si scrivono:

$$
\begin{aligned}
&\frac{\delta \sigma_r}{\delta r} + \frac{1}{r}\frac{\delta \tau_{r\theta}}{\delta \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \qquad (°)\\
&\frac{1}{r}\frac{\delta \sigma_\theta}{\delta \theta} + \frac{\delta \tau_{r\theta}}{\delta r} + 2\,\frac{\tau_{r\theta}}{r} = 0
\end{aligned}
\qquad (6)
$$

In dette equazioni R rappresenta la componente radiale della forza di massa che nel nostro caso coincide con la forza peso, agente lungo z, per cui sara' R = 0 e le (6) diventano:

$$
\begin{aligned}
&\frac{\delta \sigma_r}{\delta r} + \frac{1}{r}\frac{\delta \tau_{r\theta}}{\delta \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0\\
&\frac{1}{r}\frac{\delta \sigma_\theta}{\delta \theta} + \frac{\delta \tau_{r\theta}}{\delta r} + 2\,\frac{\tau_{r\theta}}{r} = 0
\end{aligned}
\qquad (7)
$$

Le tensioni σ_r, σ_θ, $\tau_{r\theta}$ contenute nelle (7), possono ricavarsi, con le posizioni che vedremo, mediante un'unica funzione $F\ r,\theta$ detta funzione di Airy:

$$
\begin{aligned}
&\sigma_r = \frac{1}{r}\frac{\delta F}{\delta r} + \frac{1}{r^2}\frac{\delta^2 F}{\delta \theta^2}\ ; \qquad \sigma_\theta = \frac{\delta^2 F}{\delta r^2}\\
&\tau_{r\theta} = \frac{1}{r^2}\frac{\delta F}{\delta \theta} - \frac{1}{r}\frac{\delta^2 F}{\delta r\,\delta \theta}
\end{aligned}
\qquad (8)
$$

(°) Con il simbolo δ in tutto il lavoro s'intenderà derivata parziale.

Le (8), infatti, sostituite nelle (7), le soddisfano. Il problema si riduce, allora, al calcolo della funzione $F(r, \theta)$ che deve soddisfare l'equazione di congruenza interna:

$$\left(\frac{\delta^2}{\delta r^2} + \frac{1}{r}\frac{\delta}{\delta r} + \frac{1}{r^2}\frac{\delta^2}{\delta \theta^2} \right) \cdot \left(\frac{\delta^2 F}{\delta r^2} + \frac{1}{r}\frac{\delta F}{\delta r} + \frac{1}{r^2}\frac{\delta^2 F}{\delta \theta^2} \right) = 0 \qquad (9)$$

La funzione $F(r, \theta)$ deve soddisfare la congruenza perche', se la terna di tensioni σ_{ij} e τ_{nk} non rispettasse detta equazione, noi individueremmo certamente una deformazione del sistema, ma nessuno potrebbe escludere in esso l'insorgere di vuoti o di sovrapposizione di materia. Questo in termini fisici; ragionando in termini matematici, invece, non potremmo mai essere sicuri, una volta fissata una terna di valori σ_{ij} e τ_{nk} di arrivare ad una coppia di funzioni continue ed uniformi u e v, caratterizzanti gli spostamenti (nel piano).

Per tensioni simmetriche rispetto al centro 0 (tensioni indipendenti dall'anomalia θ) le (8) diventano:

$$\sigma_r = \frac{1}{r}\frac{dF}{dr}$$

$$\sigma_\theta = \frac{d^2 F}{dr^2} \qquad (10)$$

$$\tau_{r\theta} = 0$$

espresse, ovviamente, in termini di derivate totali, per cui l'equazione (9) di congruenza interna diventa:

$$\frac{d^4F}{dr^4}+\frac{2}{r}\frac{d^3F}{dr^3}-\frac{1}{r^2}\frac{d^2F}{dr^2}+\frac{1}{r^3}\frac{dF}{dr}=0 \qquad (11)$$

che e' un'equazione differenziale ordinaria, del quarto ordine,non lineare che puo' essere trasformata in un'equazione lineare, omogenea, del quarto ordine, a coefficienti costanti, con la posizione : $r = e^t$ (12); per cui la (11) diventa:

$$\frac{d^4F}{dt^4}-4\frac{d^3F}{dt^3}+4\frac{d^2F}{dt^2}=0 \qquad (13)$$

di equazione caratteristica associata:

$$\alpha^4 - 4\alpha^3 + 4\alpha^2 = 0$$

che ammette le radici doppie 0 e 2. L'integrale generale della (13) sara' allora:

$$F = A + Bt + Ce^{2t} + Dte^{2t} \qquad (14)$$

Combinando la (14) con la (12) si ottiene:

$$F(r) = A + B\ln r + Cr^2 + Dr^2\ln r \qquad (15)$$

Se, come nel nostro caso, sono indipendenti dall'anomalia θ anche gli spostamenti, la (15) diventa:

$$F(r) = A + B\ln r + Cr^2 \qquad (16)$$

Per verificare quanto asserito basta confrontare le componenti della deformazione, espresse in coordinate polari, con le relazioni che le legano alle componenti di tensione e cioe' le:

$$\varepsilon_r = \frac{\delta u}{\delta r} \qquad ; \qquad \varepsilon_\theta = \frac{1}{r}\frac{\delta v}{\delta \theta} + \frac{u}{r}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\delta u}{\delta \theta} + \frac{\delta v}{\delta r} - \frac{v}{r} \tag{17}$$

alle

$$\begin{aligned} \varepsilon_r &= \frac{1}{E}\left(\sigma_r - \frac{\sigma_\theta}{m}\right) \\ \varepsilon_\theta &= \frac{1}{E}\left(\sigma_\theta - \frac{\sigma_r}{m}\right) \\ \gamma_{r\theta} &= \frac{1}{G}\,\tau_{r\theta} \end{aligned} \tag{18}$$

essendo G il modulo di elasticita' trasversale del materiale o seconda costante di Lame' [=] Kg/cm^2.

Dal confronto tra le (8)-(15)-(17)-(18) si ha:

$$u = \frac{1}{E}\left[-\left(\frac{m+1}{m}\right)\cdot B + 2\left(\frac{m-1}{m}\right)\cdot D \ln r - \left(\frac{m+1}{m}\right)\cdot D r + 2\left(\frac{m-1}{m}\right)\cdot C r\right] +$$

$$+ \overline{C}_1 \operatorname{sen}\theta + \overline{C}_2 \cos\theta$$

$$v = \frac{4 D r \theta}{E} + \overline{C}_3 r + \overline{C}_1 \cos\theta - \overline{C}_2 \operatorname{sen}\theta$$

Ora, affinche' gli spostamenti u e v siano indipendenti da θ deve essere:

$$D = \overline{C}_1 = \overline{C}_2 = 0$$

il che implica la validita' della (16).

Combinando la (16) con le (8) si ottiene:

$$\sigma_r = \frac{B}{r^2} + 2C = \frac{B}{r^2} + A$$

$$\sigma_\theta = -\frac{B}{r^2} + 2C = -\frac{B}{r^2} + A \qquad (19)$$

$$\tau_{r\theta} = 0$$

Per calcolare le costanti di integrazione B ed A=2C, ci serviamo delle condizioni ai limiti seguenti:

C. L. 1. per $r = \infty$ $\sigma_r = \sigma_\theta = \sigma_H$

C. L. 2 per $r = r_0$ $\sigma_r = -p_0$

Dalle (19), si ha, quindi,:

$$A = \sigma_H \quad ; \quad B = -r_0^2 \left(p_0 + \sigma_H \right) \qquad (20)$$

Alla luce di quanto calcolato, lo stato tensionale nel punto M e' pertanto:

$$\sigma_r = \sigma_H - \left(p_0 + \sigma_H \right) \cdot \left(\frac{r_0}{r} \right)^2$$

$$\sigma_\theta = \sigma_H + \left(p_0 + \sigma_H \right) \cdot \left(\frac{r_0}{r} \right)^2 \qquad (21)$$

$$\tau_{r\theta} = 0$$

Sul bordo del foro (cioe' per $r = r_0$)lo stato tensionale, allora, diventa:

$$\sigma_r = -p_0$$

$$\sigma_\theta = 2\sigma_H + p_0 \qquad (22)$$

$$\tau_{r\theta} = 0$$

Ora, affinche' σ_0 (che e' la responsabile diretta dell'apertura della fessura stessa) sia di compressione, deve essere negativa, per cui dovra' essere: $\sigma_0 = (2\,\sigma_H + p_0) < 0.$

A valle della trattazione analitica sugli sforzi di compressione da esercitare nei processi di idrofratturazione e' opportuno, per avere una chiara idea sugli ordini di grandezza delle pressioni da esercitare, estendere i calcoli fatti ad alcuni tipi di roccia. Tutto questo per poi poter confrontare tra di loro i risultati teorici ottenuti anche in relazione a prove pratiche fatte sperimentalmente in laboratorio (dall'Agip) simulando le condizioni che si verificano realmente in Natura.

A)- ROCCE GRANITICHE. Il campione preso in considerazione, da una descrizione macroscopica, risulta essere una roccia intrusiva, olocristallina, leucocratica, a grana media con quarzo e feldspati. Da un'analisi petrografica si evince che detto granito e' composto da quarzo, ortoclasio, plagioclasio, oligoclasio, biotite ed apatite come accessorio. Per tale materiale e' ν 0.2 e γ = 2.69 gr/cm^3 per cui, applicando una formula trovata in precedenza, si ha:

$$\sigma_0 = 2\,\sigma_H + p_0 = 2\left(\frac{\nu}{1-\nu}\right)\cdot\gamma\cdot h + p_0$$

cioe', nel nostro caso:

$$\sigma_0 = 2\left(\frac{0.2}{0.8}\right)\cdot\gamma\cdot h + p_0 = 0.5\cdot\gamma\cdot h + p_0$$

Per aversi uno sforzo di compressione dovra' essere:

$$|p_0| > 0.5\cdot\gamma\cdot h$$

Inserendo gli altri valori numerici, si ottiene:

$$p_0 = (0.5)\cdot(2.69)\cdot h = 1.345\ h$$

che in un piano (h, p_0) rappresenta una retta di pendenza pari a 1345. Fig.3.

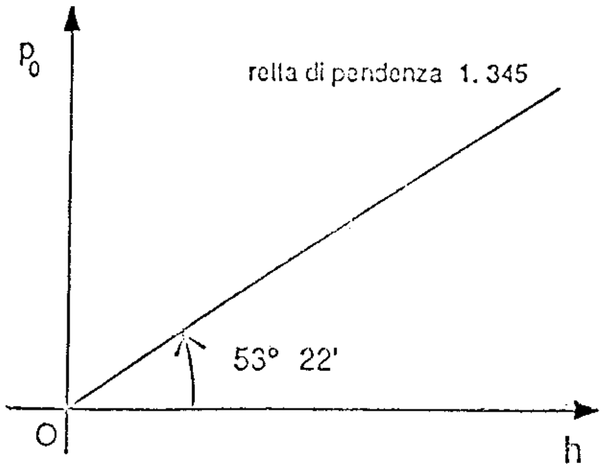

Fig. 3

Ad una profondita' h di 1000 m si ottiene:

$$p_0 = (1.345)\cdot(1000)\cdot(100) = (1.345)\cdot 10^5\ \frac{gr}{cm^2} =$$

$$= (1.345)\cdot 10^2\ \frac{Kg}{cm^2} = 134.5\ \frac{Kg}{cm^2} \cong 134\ atm.$$

B)- TUFO. Descrizione macroscopica : roccia piroclastica pomicea, di colore giallo, coerente con frammenti di rocce vulcaniche. Da un'analisi petrografica il campione in esame risulta essere un tufo giallo napoletano, costituito da pomici con cristalli di sanidino, plagioclasio ed argilla, per tale roccia e': $\nu = 0.2$ e $\gamma = 1.7\ gr/cm^3$.

Dalla solita formula si ottiene: $p_0 = (0.85)\ h$ e anche il diagramma di Fig.4.

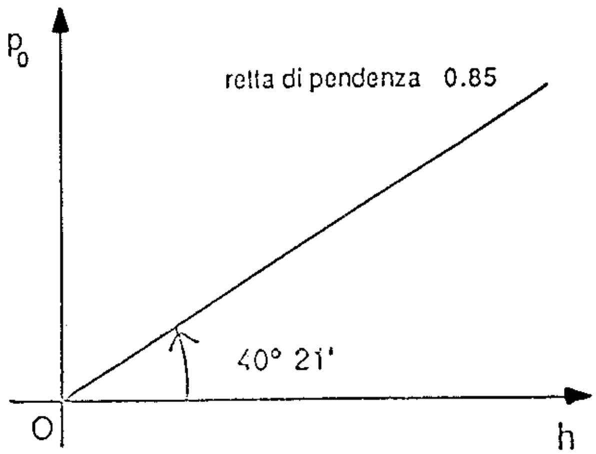

Fig.4

Alla solita profondita' h pari a 1000 m si ha:

$$p_0 = (0.85) \cdot 10^5 \frac{gr}{cm^2} = 85 \frac{Kg}{cm^2} \cong 85\ atm.$$

C)- LAVA VESUVIANA. Da una descrizione macroscopica il campione appare come una roccia effusiva compatta, di colore grigio, con evidenti cristalli di pirosseno. L'analisi petrografica ci informa che si tratta di una lava tefritica con leucite, augite e biotite.

Per essa e': $\nu = 0.2$ e $\gamma = 2.4\ gr/cm^3$.
Dai calcoli,ormai familiari, si ottiene $P_0 = (1.20)\ h$ al solito una retta rappresentata in Fig.5.

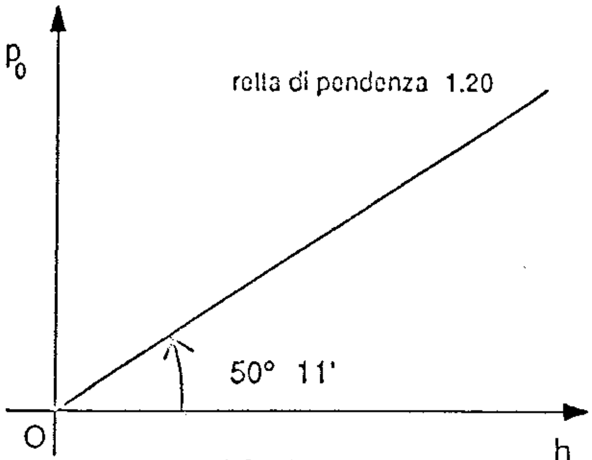

Fig.5

A 1000 m di profondita' si ha:p0 = $(1.20)\ 10^5$ g/cm^2 =120 atm.
Prove pratiche di fratturazione idraulica sono state fatte dall'Agip mineraria, sezione Geotermica, utilizzando un modello di cella per microfratturazioni, del tipo di Fig.6.

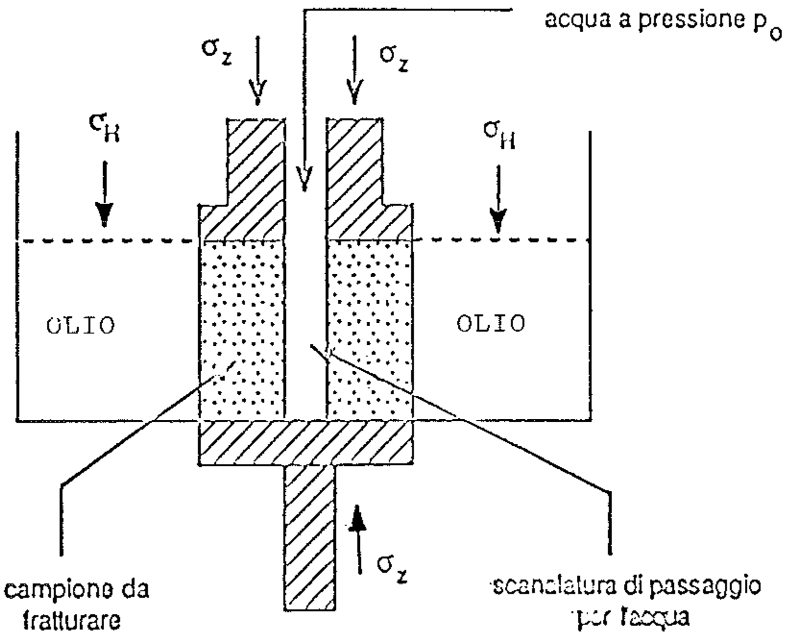

Fig.6 - Schema semplificato della cella di microfratturazione sperimentata dall'Agip.

In questo modo e' possibile applicare le pressioni σ_z e σ_H in maniera da simulare le condizioni naturali in cui il campione puo' trovarsi a diversa profondita', facendo appunto variare il valore di σ_z prendendo σ_H uguale a:

$$\sigma_H = \left(\frac{\nu}{1-\nu} \right) \cdot \sigma_z$$

Si sono ottenuti, cosi', dei valori non molto diversi da quelli calcolati analiticamente, con le equazioni prima trovate. [Verdiani G.;Moioli A. (3)]. Alla luce di quanto calcolato teoricamente e confermato in laboratorio si possono pertanto trarre le seguenti conclusioni, relativamente ai tre tipi di roccia esaminati :

a) Il granito si comporta come un materiale elastico-fragile, con un carico di rottura medio - alto, per pressioni esercitate fino a 2000 Kg/cm²; da tale valore in poi il materiale passa ad un comportamento elastico-plastico con carico di rottura molto alto.

b) Il tufo presenta un comportamento da elastico - plastico a plastico, con basso carico di rottura.

c) La lava esaminata mostra un comportamento elastico-plastico con un carico di rottura da medio-alto a molto alto.

Occorre pero' tener presente che per dare dei giudizi piu' sereni e definitivi, ci sarebbero voluti piu' campioni in esame, provenienti da diversi luoghi. La stessa influenza della temperatura, con la profondita' sulla roccia, che in laboratorio non puo' essere riprodotta, gioca nei fenomeni naturali un ruolo molto importante e delicato. Tuttavia i risultati cui si e' giunti dopo studi e calcoli analitici, da una parte e da prove di laboratorio, dall'altra, saranno certamente utili per una migliore conoscenza dello stato di fratturazione naturale, della anisotropia delle rocce, della perforabilita' delle stesse e dei sistemi di apparecchiature da adottare nelle operazioni di perforazione. Il comportamento elastico, elastico-plastico e plastico di una roccia puo' far ipotizzare, inoltre, il grado di fratturazione in cui le rocce possono trovarsi a varie profondita', potendosi avere, ancora, anche indicazioni sulla riduzione di porosita' e permeabilita' per effetto della pressione litostatica. Interpretando i risultati in termini di applicazione di formule analitiche, si possono trarre le seguenti conclusioni:

1)- Considerando, ad una medesima profondita', tre tipi di rocce con compattezza passante da una massima, attraverso una media, fino ad una meno forte, i valori individuati per le pressioni da esrcitare sulle unita' di riferimento, risultano diversi.

2)- Esaminando poi i valori derivati, in chiave di relativita', si nota come mentre sulla roccia piu' compatta occorre esercitare un tipo di pressione maggiore, la stessa automaticamente decresce in rapporto diretto alla minore compattezza della roccia.

3)- In funzione di cio', allora, sara' possibile, applicando le stesse formule, trovare per ogni tipo di roccia, il corrispondente valore della pressione, con conseguente economia di impieghi atti a produrre le relative pressioni.

BIBLIOGRAFIA (°)

1) Rau Hans, Energia Geotèrmica. ISBN.

--

2) Atti dei Convegni Lincei, Il Fenomeno Geotermico e sue applicazioni. Roma-Pisa,1975.

--

3) Verdiani G.,Moioli A.,Esplorazioni Geotermiche. AGIP ESGE.1972 1986.

--

4) Harrison E.,Kieschmick W.F.,Mc Guire W.J.,The Mechanics of Fracture Induction and Extension.1986.

--

5) Howard G.C.,Fast C.R.,Hydraulic Fracturing. Society of Petroleum Engineers of AIME.New York.1970

--

6) Rowley J.C.,Hydraulic Fragturing and Rock Mechanics Private Communication.June 1972.

--

7) Smith M.,Potter R.,Brown D.,Aamodt R.L.,Induction and Growth of Fractures in Hot Dry Rock. Stanford University Press.1973.

(°) I testi dal n.4 al n.7, riportati nella Bibliografia, e non richiamati nella trattazione, sono stati adottati dagli Autori come solo modello di studio e di riferimento per la compilazione e la stesura del lavoro.

$\mathcal{F}$-CONVERGENZA GLOBALE DEL METODO DELLE APPROSSIMAZIONI SUCCESSIVE

Nota di Giuseppina Anatriello [1]
Presentata dal Socio Antonio Zitarosa
Adunanza del 4/11/1989

Riassunto Cfr.INTRODUZIONE.

Abstract In section 3 of this paper, considering a Hausdorff space X and a continuous function f from X into itself, notion of $\mathcal{F}$-global convergence of succesive approximations method for the generalized discrete dynamical systems (X,f) is given.

In same section the question of characterization of $\mathcal{F}$-global convergence is studied.

In section 4, among other things, the question of existence of periodic points of f of a fixed period is studied.

1.INTRODUZIONE

In accordo con [8], diremo **sistema dinamico discreto generalizzato** (abbr. **s.d.d.g.**) una coppia (X,f) con X spazio di topologico e f funzione di X in sé.

(1) Borsista dell'Istituto Nazionale di Alta matematica per l'anno accademico 1989-90

Nel n.3 di questo lavoro si introduce la nozione di $\mathcal{F}$-convergenza globale del metodo delle approssimazioni successive (abbr. m.a.s.) per un s.d.d.g. (X,f) con X spazio di Hausdorff e f funzione continua.

Questa nozione generalizza quelle di convergenza globale del m.a.s. date in [2], [4], [8].

Nel n.3 si dà anche, per il s.d.d.g. (X,f), la seguente definizione, che si riferisce ad un dato sottoinsieme M di $\mathbb{N}-\{1\}$:

la coppia $(M,\mathcal{F})$ si dice **interessante** per il s.d.d.g. (X,f) se le condizioni :

A) Per (X,f) il m.a.s. $\mathcal{F}$-converge globalmente;

B) Per ogni $m\in M$, $\mathrm{Per}(f,m)=\emptyset$

sono equivalenti.

Nello stesso n.3 vengono indicati alcuni esempi di coppie interessanti e si dimostrano due teoremi di caratterizzazione della $\mathcal{F}$-convergenza globale, nel primo dei quali si suppone che una coppia $(M,\mathcal{F})$ sia interessante per (X,f^h) per ogni $h\in\mathbb{N}$, mentre nel secondo, considerato il caso $\mathcal{F}=\mathcal{P}(X)$ con X spazio di Hausdorff compatto soddisfacente il primo assioma della numerabilità, si suppone che la coppia $(\mathbb{N}/\{1\}, \mathcal{P}(X))$ sia interessante per il s.d.d.g. (X,f).

Nel n.4 di questo lavoro si indica (cfr.(4.1)) una condizione sufficiente affinché due interi positivi h e k siano tali che, per una qualunque funzione g di un insieme in sé, l'insieme dei punti periodici di g di periodo hk coincida con l'insieme dei punti periodici di g^h di periodo k.

Il teorema (4.1) permette di dimostrare, per un s.d.d.g. (X,f) tale che la coppia $(\{m\},\mathcal{F})$ sia interessante per il s.d.d.g. (X,f^h), $\forall h\in N$, due teoremi che forniscono informazioni sull'insieme dei periodi dei punti periodici di f.

Qualunque sia la funzione f di un insieme S in sé, denoteremo con f^n, $n \in \mathbb{N}$[2], l'iterata n-esima di f.

Un elemento x di S si dice **punto periodico di f di periodo k**, con $k \in \mathbb{N}$, quando risulta:

$$f^k(x)=x, \quad f^i(x) \neq x \quad \forall i \in \{1,\dots,k\}/\{k\};$$

col simbolo:

$$\mathrm{Per}(f,k)$$

si denota l'insieme dei punti periodici di f di periodo k.

Evidentemente:

$$\mathrm{Per}(f,1)=\mathrm{Fix}f,$$

dove Fixf è l'insieme dei punti fissi di f.

Faremo uso delle seguenti proposizioni (vedi (1.1) e (1.2) di [5]):

(2.1). Qualunque sia l'intero positivo n, condizione necessaria e sufficiente affinché un elemento x di S sia un punto fisso di f^n è che x sia un punto periodico di f e il suo periodo sia un divisore di n.

(2.2). Qualunque siano gli interi positivi m e n, posto:

$$d=\mathrm{M.C.D.}(m,n),$$

risulta:

$$\mathrm{Per}(f,n) \subseteq \mathrm{Per}(f^m, n/d)$$

Se (S,f) è un s.d.d.g., un sottoinsieme non vuoto T di S è detto **insieme positivamente invariante** (risp. **invariante)** di (S,f) se $f(T) \subseteq T$ (risp. $f(T)=T$) (vedi [9]).

Un insieme invariante chiuso I di (S,f) è detto **insieme minimale** di (S,f) se non esiste un sottoinsieme di I che sia un insieme chiuso invariante di (S,f) (vedi [9]).

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(2) Con $\mathbb{N}$ denotiamo l'insieme degli interi positivi e con $\mathbb{N}_0$ l'insieme $\mathbb{N} \cup \{0\}$.

Richiamiamo la seguente proposizione inclusa nella (3.2) di [9]:

(2.3). Se S è uno spazio compatto di Hausdorff soddisfacente il primo asioma della numerabilità e f è una funzione continua di S in sè, si ha che:

a) il s.d.d.g. (S,f) è dotato di un insieme minimale;

b) un sottoinsieme non vuoto Y di S è un insieme minimale del s.d.d.g. (S,f) se e solo se:

$$\Omega(y)=Y,\ \forall y\in M\ (^3).$$

3.$\mathscr{F}$-CONVERGENZA GLOBALE.

D'ora in poi indicheremo con X uno spazio di Hausdorff, con $\mathscr{F}$ un insieme di parti di X a cui appartengano quelle finite e con f una funzione continua di X in sé.

Diremo che per il s.d.d.g. (X,f) il metodo delle approssimazioni successive **$\mathscr{F}$-converge globalmente** se, per ogni $x\in X$ tale che il codominio della successione $(f^n(x))_{n\in\mathbb{N}}$ appartenga $\mathscr{F}$, la successione stessa converge (a un punto fisso di f).

E' facile verificare che:

(3.1). Se per il s.d.d.g. (X,f) il m.a.s. $\mathscr{F}$-converge globalmente, allora:

$$\mathbf{Per(f,k)=\emptyset,\ \forall k\in\mathbb{N}-\{1\}.}$$

DIM. Siano per assurdo $\bar{x}\in X$ e $h\in\mathbb{N}-\{1\}$ tali che:

$$(*)\quad \bar{x}\in \mathrm{Per}(f,h),$$

allora la successione $(f^n(\bar{x}))_{n\in\mathbb{N}}$, avendo codominio finito e perciò appartenendo a $\mathscr{F}$, converge.

Ma ciò è assurdo in quanto lo spazio X è di Hausdorff e le sucessioni estratte:

(3) Con $\Omega(y)$ denotiamo l'insieme dei punti limite della successione $(f^n(y))_{n\in\mathbb{N}}$.

$$(f^{nh}(\overline{X}))_{n\in\mathbb{N}},\quad (f^{n(h+1)}(\overline{X}))_{n\in\mathbb{N}},$$

convergono per la (*) a punti distinti.

L'asserto è così dimostrato.

Se X è un insieme totalmente ordinato munito dell'order topology (risp. uno spazio metrico) e $\mathcal{F}$ è l'insieme delle parti limitate di X, la $\mathcal{F}$-convergenza globale coincide con la "convergenza globale" definita in [4] (risp. [8]).

Se:

$$X=\{z\in\mathbb{C} \,/\, |z|=1\}$$

la $\mathcal{P}(X)$-convergenza globale coincide con la "convergenza globale" definita in [2].

Se X è un albero connesso per archi la $\mathcal{P}(X)$-convergenza globale non è altro che la "convergenza globale" definita in [2].

D'ora innanzi con M indicheremo un sottoinsieme di $\mathbb{N}/\{1\}$.

Diremo che la coppia $(M,\mathcal{F})$ è **interessante** per il s.d.d.g. (X,f) se le condizioni:

A) Per (X,f) il m.a.s. $\mathcal{F}$-converge globalmente.

B) Per ogni $m\in M$, $Per(f,m)=\emptyset$,

sono equivalenti.

Diamo alcuni esempi di coppie interessanti.

1. Se X è un insieme totalmente ordinato, completo e denso in sé, munito dell'order topology e $\mathcal{F}$ è l'insieme delle parti limitate di X, la coppia $(\{2\},\mathcal{F})$ è interessante per $(X,\mathcal{F})$ (vedi [4]).

2. Se:

$$X=\{z\in\mathbb{C} \,/\, |z|=1\}$$

e M è l'insieme degli interi positivi pari, la coppia $(M,\mathcal{P}(X))$ è interessante per (X,f) (vedi [2]).

3. Se X è un albero connesso per archi dotato di un numero finito, μ, di punti terminali, la coppia $(\{2,\dots,\mu\},\mathcal{P}(X))$ è interessante per (X,f) (vedi [6]).

4. Se sono verificate le condizioni:

a) X è un albero connesso per archi tale che l'insieme dei suoi punti terminali è numerabile,

b) per ogni $x \in \mathrm{Fix} f$, $X-\{x\}$ è dotato di un numero finito di componenti,

la coppia $(\mathbb{N}/\{1\}, \mathcal{P}(X))$ è interessante per (X,f) (vedi [3]).

5. Se $X=[0,1]^2$ e:

$$f(x,y)=(g(x,y),x), \ \forall (x,y) \in X,$$

la coppia $(\{4\}, \mathcal{P}(X))$ è interessante per (X,f) se f è decrescente rispetto ad entrambe le variabili e inoltre, posto:

$$\varphi(x)=g(x,x), \ \forall x \in [0,1],$$

α) $(\varphi(x)-x)(\varphi^2(x)-x) \geq 0, \ \forall x \in [0,1],$

β) non esiste un punto periodico ξ di φ di periodo 2 ed un punto $P \in [0,1]^2$ tale che $(\xi, \varphi(\xi))$ sia un punto limite della successione $(f^n(P))_{n \in \mathbb{N}}$ (vedi [7]).

Passiamo ora a dimostrare il seguente teorema di caratterizzazione della $\mathcal{F}$-convergenza globale:

(3.2). Se $(M, \mathcal{F})$ è una coppia interessante per (X, f^h), $\forall h \in \mathbb{N}$, allora le seguenti condizioni sono equivalenti:

i) Il m.a.s. $\mathcal{F}$-converge globalmente per (X,f).

ii) Per ogni $h \in \mathbb{N}$ c'è la $\mathcal{F}$-convergenza globale del m.a.s per (X, f^h).

iii) Esiste un intero positivo h', non multiplo di alcun $m \in M$, tale che ci sia la $\mathcal{F}$-convergenza globale del m.a.s per $(X, f^{h'})$.

DIM. i) $\Rightarrow$ ii). Supponiamo per assurdo che la ii) non sia verificata. Allora, tenendo presente la (3.1), si ha l'esistenza di un intero h, di un $m \in M$ e di un $x \in X$ tali che:

$$x \in \mathrm{Per}(f^h, m) \subseteq \mathrm{Fix} f^{hm}.$$

Conseguentemente, per la (2.1), esiste $m' \geq 1$ tale che:

$$x \in Per(f,m').$$

Il caso $m'=1$ è escluso poiché x non può essere un punto fisso di f^h, e d'altro canto il caso $m'>1$ è, per la (3.1), in contrasto con la i).

ii) $\Rightarrow$ iii). Evidente.

iii) $\Rightarrow$ i). Per la (3.1) la iii) implica che:

$$Per(f^{h'},r)=\emptyset, \ \forall r \in \mathbb{N}-\{1\};$$

allora, posto:

$$d_n = M.C.D.(h',n), \ \forall n \in \mathbb{N},$$

essendo:

$$m/d_m > 1, \ \forall m \in M,$$

si ha:

$$Per(f^{h'},m/d_m)=\emptyset, \ \forall m \in M.$$

Conseguentemente, poiché per la (2.2) risulta:

$$Per(f,m) \subseteq Per(f^{h'},m/d_m), \ \forall m \in M,$$

si ha:

$$Per(f,m)=\emptyset, \ \forall m \in M.$$

e la i) è verificata.

Il teorema è così completamente dimostrato.

Considerando gli esempi 1 e 3, dal teorema (3.2) si ricava in modo ovvio che:

(3.3). Se X è un insieme totalmente ordinato, completo e denso in sé, munito dell'order topology (risp. un albero connesso per archi dotato di un numero finito, μ, di punti terminali) e $\mathcal{F}$ è l'insieme delle parti limitate di X (risp. delle parti di X), la i) e la ii) del teorema (3.2) sono equivalenti alla condizione:

j) esiste un intero positivo dispari h' (risp. un intero positivo h' non multiplo di alcun numero dell'insieme $\{2,\ldots,\mu\}$) tale che ci sia la $\mathcal{F}$-convergenza globale del m.a.s. per $(X,f^{h'})$.

Relativamente a tale proposizione, notiamo che essa è stata già provata in [1], limitatamente alla equivalenza tra la i) e la j).

Un'altra applicazione della (3.2) si ottiene considerando l'esempio 2.

Dimostriamo infine il seguente teorema di caratterizzazione della $\mathcal{P}(X)$-convergenza globale:

(3.4). Se X è uno spazio compatto soddisfacente il primo assioma della numerabilità e $(\mathbb{N}-\{1\},\mathcal{P}(X))$ è una coppia interessante per (X,f), le seguenti condizioni sono equivalenti:

i) Ogni insieme minimale di (X,f) è un singleton.

ii) Ogni insieme minimale finito di (X,f) è un singleton.

iii) Per (X,f) il m.a.s. $\mathcal{P}(X)$-converge globalmente.

iv) Per ogni $x \in X$, la successione $(f^n(x))_{n\in\mathbb{N}}$ è dotata di un punto limite appartenente a Fixf.

DIM. i) $\Rightarrow$ ii). Evidente.

ii) $\Rightarrow$ iii). Per la ii), infatti, risulta:

$$Per(f,n)=\emptyset, \quad \forall n\in\mathbb{N}.$$

e quindi la iii) è soddisfatta.

iii) $\Rightarrow$ iv). Evidente.

iv) $\Rightarrow$ i). Se Y è un insieme minimale di (X,f), per la (2.3) si ha:

$$Y = \Omega(y), \quad \forall y\in Y,$$

e quindi per la iv) esiste un $\overline{y}\in Y$ tale che:

$$\overline{y}\in Fixf;$$

ma allora si ha:

$$Y=\Omega(\overline{y})=\{\overline{y}\}.$$

L'asserto è così completamente dimostrato.

Utilizzando la (2.1) e la (2.2) proviamo che:

(4.1). Se g è una funzione di un insieme in sé e gli interi positivi h e k sono tali che:

$$\mathbf{M.C.D.(r,k)>1\ \forall r>1\ /\ r|h,}$$

risulta:

$$\mathbf{Per(g,hk)=Per(g^h,k).}$$

DIM. Per la (2.2) basta evidentemente provare che:

$$Per(g^h,k)\subseteq Per(g,hk).$$

Sia x un elemento di $Per(g^h,k)$; poiché x è un punto fisso di g^{hk}, esso è per la (2.1) un punto periodico di g di periodo:

$$h'k', \text{con } h'|h \text{ e } k'|k.$$

Proviamo che necessariamente deve essere:

$$k'=k \quad e \ h'=h.$$

Innanzi tutto, essendo $h'|h$, risulta:

$$x=g^{h'k'}(x)=g^{hk'}(x)$$

e quindi, per l'appartenenza di x a $Per(g^h,k)$, si ha:

$$k'=k.$$

Supponiamo ora per assurdo che:

$$h=rh' \text{ con } r>1;$$

allora, posto:

$$s=M.C.D.(r,k),$$

si ha:

$$s>1, \quad x=g^{h'k}(x)=g^{h'sk/s}(x),$$

e poiché evidentemente $h's|h$ si ha pure:

$$x=g^{hk/s}(x),$$

contro l'ipotesi che il periodo di x rispetto a g^h è k.

L'asserto è così dimostrato.

Dalla (4.1) consegue banalmente che:

(4.2). Se g è una funzione di un insieme in sé, qualunque siano gli interi $n,s\in\mathbb{N}$ e $r\in\mathbb{N}_0$, risulta:

$$\mathbf{Per}(g, n^{r+s}) = \mathbf{Per}(g^{n^r}, n^s).$$

La proposizione (4.2) generalizza la (1.4) di [5], che si riferisce al caso in cui n è primo.

(4.3). Se l'intero positivo $m>1$ è tale che la coppia $(\{m\}, \mathcal{F})$ sia interessante per (X, f^k), $\forall k \in \mathbb{N}$, allora, per ogni $t \in \mathbb{N}$ tale che:

$$\mathrm{Per}(f, m^t) \neq \emptyset,$$

risulta:

$$\mathrm{Per}(f, m^i) \neq \emptyset, \quad \forall i \in \{1, \ldots, t\}.$$

DIM. Per la (4.2) si ha che:

$\mathrm{Per}(f, m^t) = \mathrm{Per}(f^{m^{i-1}}, m^{t-i+1})$, $\mathrm{Per}(f, m^i) \subseteq \mathrm{Per}(f^{m^{i-1}}, m)$; conseguentemente, poiché per la (3.1) sussiste l'implicazione:

$$\mathrm{Per}(f^{m^{i-1}}, m^{t-i+1}) \neq \emptyset \Rightarrow \mathrm{Per}(f^{m^{i-1}}, m) \neq \emptyset,$$

l'asserto è dimostrato.(4)

(4.4). Se l'intero positivo $m>1$ è tale che la coppia $(\{m\}, \mathcal{F})$ sia interessante per (X, f^k), $\forall k \in \mathbb{N}$, allora, qualunque siano l'elemento n di $\mathbb{N}$ e l'intero $l>1$ tali che:

$$\mathrm{M.C.D.}(l, m) = 1, \quad \mathrm{Per}(f, m^n l) \neq \emptyset,$$

risulta:

$$\mathrm{Per}(f, m^i) \neq \emptyset, \quad \forall i \in \mathbb{N}.$$

DIM. Per la (4.3) basta dimostrare che qualunque sia l'intero $h>n$ si ha:

$$\mathrm{Per}(f, m^h) \neq \emptyset.$$

Ora, per la (2.2), risulta:

$$\mathrm{Per}(f, m^n l) \subseteq \mathrm{Per}(f^{m^{h-1}}, l)$$

e quindi $\mathrm{Per}(f^{m^{h-1}})$ è non vuoto; conseguentemente, per la (4.2) e la (3.1) si ha:

$$\mathrm{Per}(f, m^h) = \mathrm{Per}(f^{m^{h-1}}, m) \neq \emptyset.$$

L'asserto è così dimostrato.(5)

(4) Cfr., anche per la dimostrazione, la proposizione (2.3) di [5].

(5) Cfr., anche per la dimostrazione, la proposizione (2.4) di [5].

BIBLIOGRAFIA

[1] G. ANATRIELLO, "Sistemi dinamici discreti generalizzati: insiemi minimali, punti periodici, convergenza globale"; Tesi di laurea (1989).

[2] G. DI LENA, "Global convergence of the method of approximations on S^1";J.of Math. Anal. and Applic., vol.106,n.1,p.196-201 (1985).

[3] G. DI LENA, B. MESSANO, "Generalized discrete dynamical systems: minimal sets, periodic points, global convergence"; Preprint (1989).

[4] B. MESSANO, "Convergenza globale del metodo delle approssimazioni successive in un insieme totalmente ordinato"; Rend. di Mat., (4), vol.2, serie VII, p.725-739 (1982).

[5] B. MESSANO, "Sull'esistenza di punti periodici di una applicazione di un insieme totalmente ordinato in sé"; Le Matematiche, vol. XXXV, fasc. I-II (1980), p. 287-300 (1983).

[6] B. MESSANO, "Contnuous functions from an arcwise connected tree into itself: periodic points, global convergence, plus-global convergence"; in corso di stampa su Ricerche di Matematica.

[7] B. MESSANO, "Global convergence and non existence of periodic points of period 4"; Preprint (1989).

[8] B. MESSANO, A. ZITAROSA "Sul metodo delle approssimazioni successive: convergenza globale e plus-convergenza globale"; Rend. del Circ. Mat. di Palermo serie II, tomo XXXVII, p.246-260 (1988).

[9] B. MESSANO, A. ZITAROSA "Minimal sets of generalized dynamical systems"; Preprint (1989).

20 EPI-24-METHYLENLOPHENOL
A NEW STEROL WITH ANTIALGAL ACTIVITY[1]

Nota di P.MONACO, L.PREVITERA* e del Socio L.MANGONI[2]

Adunanza del 2/12/1989

Riassunto - La struttura di 20 epi 24-metilenlofenolo (20S-4α-metil-24- metilen-colest-7-en-3ß-olo), definita su basi spettroscopiche e per correlazione chimica, é stata attribuita ad uno sterolo con attività antialgale isolato dalla *Typha latifolia*.

Summary- An antialgal sterol with an unusual 20S configuration has been isolated from *Typha latifolia*. The structure 20 epi 24-methylenlophenol (20S-4α-methyl-24-methylen-cholest-7en-3ß-ol) has been defined by spectroscopic studies and chemical correlation.

Allelochemistry in aquatic systems has not been much studied. The few data are mainly related to the detrimental effects of algae on aquatic macrophytes (1) and to the ability of algae to secrete substances that inhibit predecessor species and stimulate successor species (2).

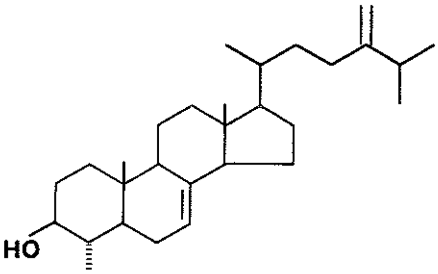
In connection with our interest in the effect of higher aquatic plants in controlling phytoplankton in eutropic systems (3) we have now examined *Typha latifolia* L., a plant used in traditional Chinese medicine as a remedy for bloody stools.

The ethereal extract of *Typha latifolia* showed activity against several blue-green algae in a paper disk bioassay (3). The activity was monitored during all the chromatographic separations until a pure active compound was obtained.

[1]A full paper on this subject has been submitted to Phytochemistry.

[2]Dipartimento di Chimica Organica e Biologica dell'Università di Napoli.

Structure 1 of (20S)-4α-methyl-24-methylen-cholest-7-en-3ß-ol was attributed to this antialgal sterol by spectroscopic evidences and chemical correlation.



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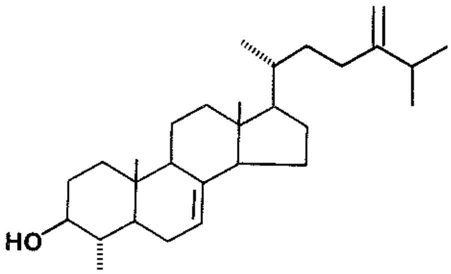

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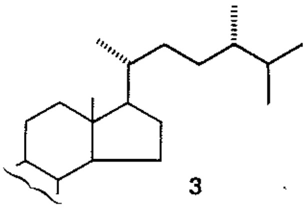
Compound 1 had m.p. 176-78°, $[\alpha]_D$ +3° and a high resolution IEMS spectrum with a molecular ion at m/z 412.3681 corresponding to $C_{29}H_{48}O$ and fragments at m/z 397.3450, 328.2739, 287.2320, 285.2190 and 260.2166. ^{1}H-nmr spectrum showed the H-18 and the H-19 methyls as singlets at δ 0.546 and 0.836 respectively, four methyl doublets centred at δ 0.985, 0.997, 1.032 and 1.037 attributed to H-21, 4α-Me, H-26 and H-27 respectively, the H-3 methyne as a multiplet at δ 3.131, the vinylic H-7 proton as a multiplet at δ 5.182 and finally the H-28 methylene as an AB system at δ 4.662 and 4.709. These data were identical with those of 24-methylenlophenol (2), m.p. 172-73°, $[\alpha]_D$ +6° we have recently isolated from the green alga *Dunaliella acidophila* (4) with the exception of the H-21 methyl resonance which was 0.024 ppm downfield in 1. By analysis of several solutions of 1 and 2 such a difference in the chemical shift was shown to be indipendent of the samples concentration.

Also the ^{13}C-nmr spectra of 1 and 2 (4) showed no remarkable differences being only C-18 0.41 ppm upfield and C-22 0.34 ppm downfield in this latter.

Significative differences between 1 and 2 were instead found in the proton proton NOE differences spectroscopy and in the biological properties :in fact NOE to H-21 methyl from the H-18 one was present only in 24-methylenlophenol (2) as well as when a sample of 2 was checked for its antialgal activity in the paper disk bioassay no trace of activity was monitored.

All these data suggested that **1** could have an inverted configuration at C-20 even if in a previous paper on 20 epi-cholesterol (5) as well as in a paper on different 24-methyl-sterols (6) a 0.09 ppm upfield shift of H-21 was reported for the 20S stereomers. As this difference could plausibly be attributable to the presence of an unsaturation in the side chain, 1 and 2 were hydrogenated in neutral medium: **2** gave an 1:1 mixture of (20R,24S)-4α,24-dimethyl-cholest-7-en-3ß-ol (**3**) (7) and (20R,24R)-4α,24-dimethyl cholest-7-en-3ß-ol (**4**) whereas hydrogenation of **1** gave an 1:1 mixture of two sterols whose data justify the attribution of the epimeric (20S,24S) (**5**) and (20S,24R) (**6**) structures.



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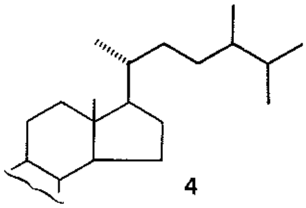

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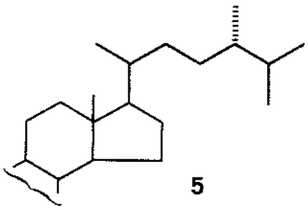

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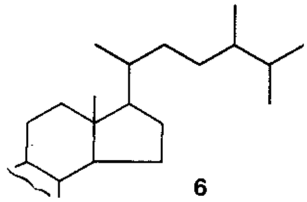

6

A comparison of high field ^{1}H-nmr spectra of **3** and **5** (**4** and **6**) showed that H-21 was about 0.1 ppm upfield in the isomeric (20S) sterols according to the previous mentioned data thus confirming the structure **1**.

REFERENCES

1) Sharma,K.P. (1985) *Aquat. Botany* **22**, 71.
2) Keating,K., (1977) *Science* , **196**, 885.

3) Della Greca,M., Monaco,P., Previtera,L., Aliotta,G., Pinto,G. and Pollio,A. (1989) *Phytochemistry*, **28** 2319.
4) Pollio,A., Della Greca,M., Monaco,P., Pinto,G. and Previtera,L. (1988) *Biochim.Biophys.Acta*, **963**, 53.
5) Nes,W.R., Varkey,T.E. and Krevitz,K. (1977) *J.Am.Chem.Soc.*, **99** 260.
6) Kokke,W.C.M., Bohlin,L., Fenical,W. and Djerassi,C. (1982) *Phytochemistry*, **21** 881.
7) Koreeda,M. and Koizumi,N. (1978) *Tetrahedron Letters* **19** 1641.

ON THE EXISTENCE AND STABILITY OF A STEADY SOLUTION TO A NONLINEAR INTEGRAL EQUATION OF THE PARTICLE TRANSPORT THEORY

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Riassunto. Si dimostra un teorema di esistenza ed un teorema di stabilita' asintotica per la soluzione stazionaria di un'equazione integrale non lineare interessante la teoria del trasporto delle particelle nell'ambito dei modelli stocastici.

Abstract. The existence and asymptotic stability of the steady solution to a nonlinear integral equation of the particle transport theory in the stochastic models is proved.

1. Introduction.

The determination of the evolution of the one particle distribution function $f(\mathbf{x},\mathbf{v},t)$ in a gas of particles diffusing in an infinite homogeneous medium through binary collisions, has drawn the attention of many researchers. The Boltzmann equation which determines $f(\mathbf{x},\mathbf{v},t)$ is an integro-differential equation and has the following standard form [4,11] :

$$(1)\qquad \left(\partial_t+\mathbf{v}\cdot\nabla_x+(F/m)\cdot\nabla_v\right)f(\mathbf{x},\mathbf{v},t)=C(f,f)$$

where m is the mass particle, $\mathbf{F}$ the external force and $C(f,f)$ is the collisional term that has been represented in various different version. In this paper $C(f,f)$ is supposed to satisfy the so-called scattering kernel formulation [1]. Moreover we assume that :

a) $\mathbf{F}=0,\nabla_x f(\mathbf{x},\mathbf{v},t)=0$

b) The cross section obey the k/v-law (k positive constant).

c) At time $t=0$, Q particles (per unit volume) are injected with the velocity distribution $S(v)$ by a pulsed source $Q^*=QS(v)\delta(t)$ [1] .

Then, it is has been shown that [9-10] equation (1) is equivalent to

$$
(2)\quad \begin{cases} f_t+kQf=k\displaystyle\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}\pi(v',v'',v)f(v',t)f(v'',t)dv'dv'' \\[2em] f(v,0)=QS(v) \qquad\qquad v,v',v''\in\mathbb{R}_3 \quad , \quad t\in[0,\infty) \end{cases}
$$

In (2)

$$
(3)\qquad Q=\int_{\mathbb{R}_3} f(v,t)dv
$$

represent the number of particles injected in the medium while v',v'',v are the velocities before and after the collisions respectively. The function $\pi(v',v'',v)$ is the scattering probability distribution which is a non negative function on $R_3\times R_3\times R_3$, is summable respect to v and obeys respectively the normalization and symmetry conditions:

$$
(4)\int_{\mathbb{R}_3}\pi(v',v'',v)dv=1 \qquad , \qquad \pi(v',v'',v)=\pi(v'',v',v)
$$

Equation (2) can be integrated along the trajectory of the particle to yield :

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[1] The symbol $\delta(t)$ represent the Dirac delta function. Whithout loss, of generality we adopt the normalization $\int_{\mathbb{R}_3}S(v)dv=1$.

$$(5)\quad f(v,t)=QS(v)e^{-kQt}+ke^{-kQt}\int_0^t e^{-kQs}ds\int_{\mathbb{R}3}\int_{\mathbb{R}3}\pi(v',v'',v)\times$$
$$f(v',s)f(v'',s)dv'dv''$$

The problem considered, has been, in the last years, object of several investigations aimed at focusing the mathematical problems connected with the existence, uniqueness and stability of the solutions. We recall that in [2-3-6] the global existence and uniqueness of solutions to equation (2) has been given in the natural space $L^1(R3)\times[0,\infty)$. In [7-8] has been studied essentially the stability of the solutions to equation (2) (and some implications) respectively in the space $L^2\times[0,\infty)$ and in the natural space $L^1\times[0,\infty)$. Let us recall that - in the case of Maxwellian particles with a cut-off - for a spherically symmetric interaction potential and for deterministic models based on momentum and energy conservation laws, the homogeneous usual Boltzmann equation is equivalent to equation (2) with a suitable specified π [5,9-10]. But when one considers a stochastic model, then π is unspecified.

In the present paper - in the framework of stochastic models for which π is unspecified - we assume that the scattering probability distribution π is expanding as power series of a parameter ε and we define an iterative constructive scheme leading to explicit solution of the form of power series of ε. We then prove the existence of a steady solution to equation (2) and its L^1-asymptotic stability.

The paper is organized as follows. In Sect.2 we introduce an iterative constructive scheme for the solution to equation (2) while in Sect.3 we prove the existence of a steady solution. Sects.4,5 are respectively dedicated to the L^1-attractivity of the steady solution and its L^1-asymptotic stability.

2. Solutions.

It is easily seen that when the scattering probability distribution is independent of the velocities before the collision i. e.

(6) $$\pi(v',v'',v)=\pi_0(v) \quad , \qquad \int_{\mathbb{R}^3}\pi_0(v)dv=1$$

The solution to equation (2) is [2]:

(7) $$f_0(v,t)=Q[\pi_0(v)+(S(v)-\pi_0(v))e^{-kQt}]$$

In the sequel we shall assume:

Hypothesis I. The scattering probability distribution is expanding about $\pi_0(v)$ as power series of a parameter ε, not necessarily small, according to

(8) $$\pi(v',v'',v)=\pi_0(v)+\sum_{i=1}^{\infty}\varepsilon^i\pi_i(v',v'',v)$$

In order to satisfy equations (4) we require

(9) $$\int_{\mathbb{R}^3}\pi_i(v,v'',v)dv=0 \quad , \quad \pi_i(v',v'',v)=\pi_i(v'',v',v)$$

It is then quite natural to seek a solution of equation (2) in the form

(10) $$f(v,t)=f_0(v,t)+\sum_{i=1}^{\infty}\varepsilon^i f_i(v,t)$$

with:

(11) $$\int_{\mathbb{R}^3}f_i(v,t)=0 \quad , \qquad f_i(v,0)=0$$

It is easy to verify that consequently the evolution equations for f_i $(i=1,2,..)$ are

(12) $$\frac{\partial f_1}{\partial t}+kQf_1=k\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\pi_1(v',v'',v)f_0(v',t)f_0(v'',t)dv'dv''$$

$$(13)\quad \frac{\partial f_2}{\partial t} + kQf_2 = k\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}[\pi_2(v',v'',v)f_0(v',t)f_0(v'',t) +$$

$$+ 2\pi_1(v',v'',v)f_0(v',t)f_1(v'',t)]dv'dv''$$

$$(14)\quad \frac{\partial f_3}{\partial t} + kQf_3 = k\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}[\ \pi_3(v',v'',v)f_0(v',t)f_0(v'',t)+$$

$$+2\pi_1 f_0(v',t)f_2(v'',t)+2\pi_2(v',v'',v)f_0(v',t)f_1(v'',t)+$$

$$+\pi_1(v',v'',v)f_1(v',t)f_1(v'',t)]dv'dv''$$

- -

i.e.,accounting for the $(9)_2$

$$(15)\qquad \frac{\partial f_i}{\partial t} + kQf_i = k\sum_{\substack{s+k+l=i}}^{s,k,l\in N}\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}\pi_s(v',v'',v)f_k(v',t)\times$$

$$f_l(v'',t)dv'dv''. \qquad (i=1,2,3,\ldots)$$

Recalling that π_i ($i=0,1,2..$) are known, from equation (12) one obtains $f_1(v,t)$.Inserting $f_1(v,t)$ in (13) one obtain $f_2(v,t)$ and so on.

Now we are in position to prove the following theorem:

Theorem 1. If $\exists$ $M>0$ such that:

$$(16)\qquad \left|\int_{\mathbb{R}_3}\pi_n(v',v'',v)\varphi_1(v')\varphi_2(v'')dv'dv''\right|\le M^n$$

$$\forall\ \varphi_i(v)\ :\ \int_{\mathbb{R}_3}\varphi_i(v)dv \le Q \qquad i=1,2$$

and moreover holds (8) with $\varepsilon<1/M$,then the series

$$(17)\qquad f(v,t)=f_0(v,t)+\sum_{i=1}^{\infty}\varepsilon^i f_i(v,t) \qquad ,\ \varepsilon<1/M$$

(where $f_i(v,t)$ are L^1-solutions to the initial value problem (15)-$(11)_2$),is a.e. uniformly convergent in $\mathbb{R}_3\times[0,\infty]$.Moreover (17) can be differentiated (respect to t) term by term, is termwise L^1-summable (respect to **v**) and is solution to equations (2).

Proof. By equation (15) and the hypothesis of the theorem we obtain:

$$(18) \qquad |f_n(\mathbf{v},t)| \le n^2 M^n/Q$$

Then for $\varepsilon<1/M$ series (17) ,accordingly to the Weierstrass'test,is a.e. uniformly convergent in $\mathbb{R}_3\times[0,\infty]$. Moreover by equation (15) we obtain

$$(19) \qquad \left|\frac{\partial f_n(\mathbf{v},t)}{\partial t}\right| \le 2kn^2M^n$$

hence derivative series is uniformly convergent and (17) can be differentiated,respect to t,term by term.Finally one verify easily that this series is termwise L^1-summable (respect to v) and is solution to equation (2).

3. Existence of the steady solution.

Let us consider now the steady version of equation (2):

$$(20) \qquad f(\mathbf{v})=(1/Q)\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}\pi(\mathbf{v}',\mathbf{v}'',\mathbf{v})f(\mathbf{v}')f(\mathbf{v}'')d\mathbf{v}'d\mathbf{v}''$$

We are now in position to prove the following theorem:

Theorem 2. Let assumptions of theorem 1 hold.Then exists a solution to equation (20).

Proof. Let us consider the function

$$(21) \qquad M(\mathbf{v})=M_0(\mathbf{v})+\sum_{i=1}^{\infty}\varepsilon^i M_i(\mathbf{v})$$

where

$$(22) \qquad M_0(\mathbf{v})=Q\pi_0(\mathbf{v})$$

$$(23)\qquad M_1(v)=(1/Q)\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}\pi_1(v',v'',v)M_0(v')M_0(v'')dv'dv''$$

$$(24)\qquad M_2(v)=(1/Q)\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}[\pi_2(v',v'',v)M_0(v',t)M_0(v'',t)\ +$$

$$+\ 2\pi_1(v',v'',v)M_0(v',t)M_1(v'',t)]dv'dv''$$

- -

Substituting (21) in (20) and taking into account theorem 1 follows that series (21) is a.e. uniformly convergent in $\mathbb{R}_3\times[0,\infty]$, termwise L^1-summable (respect to v) and is solution to equation (20).

4. Attractivity of the steady solution.

Theorem 3. Let us assume the validity of theorem 1.If $\exists$ $\Phi_i(v',v'',v)\in L^1(R_6)$ $(\forall v)$,such that:

$$(25)\qquad |\pi_i(v',v'',v)\Psi_1(v')\Psi_2(v'')|\le\Phi_i(v,v'',v)$$

$$\forall\ \Psi_i(v)\ :\ \int_{\mathbb{R}_3}\Psi_i(v)dv\ \le\ Q$$

then indicated by $f(v,t)$ a solution to equation (2) exists the limit:

$$(26)\qquad \lim_{t\to\infty} f(v,t)=M(v)\qquad \text{a.e.}$$

and does not depend on the initial data.

Proof. Integrating equation (12) and taking into account the initial condition $f_1(v,t)=0$, we have:

$$(27)\qquad f_1(v,t)=ke^{-kQt}\int_0^t e^{-kQs}F_1(s)ds$$

where

$$(28)\qquad F_1(t)=\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}\pi_1(v',v'',v)f_0(v',t)f_0(v'',t)dv'dv''.$$

But equation (7) implies:

(29) $$\lim_{t\to\infty} f_0(v,t)=Q\pi_0(v)$$

and by equation (27) we easily obtain

(30) $$\lim_{t\to\infty} f_1(v,t)=\lim_{t\to\infty} F_1(t)/Q$$

Now for the hypothesis of the theorem is possible to apply the Lebesque's theorem and to obtain

(31) $$\lim_{t\to\infty} F_1(t)=Q^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\pi_1(v',v'',v)\pi_0(v')\pi_0(v'')dv'dv''= =H_1(v).$$

Analogously for the other terms $f_i(v,t)$, we easily obtain

(32) $$\exists\ H_i(v)\ :\quad \lim_{t\to\infty} f_i(v,t)=H_i(v) \quad \text{a.e. } i=2,3,\ldots$$

then, taking into account theorem 1, follows:

(33) $$\lim_{t\to\infty} f(v,t)=M(v) \quad \text{,a.e.}$$

and the theorem is completely proved.

Let f(v,t) and g(v,t)=f+u be two solutions to equation (2) corresponding respectively to the initial data $(2)_2$ and

(34) $$g(v,0)=QS(v)+u_0(v).$$

Indicated by

(35) $$\|u\|=\int_{\mathbb{R}^3}|u(v,t)|dv$$

the L^1-norm of the perturbation u(v,t) to the basic solution,let us recall the definition of unconditional attractivity.

Definition 1. The solution f(v,t) is called unconditionally attractive if

$$(36) \qquad \lim_{t\to\infty} \|u(v,t)\|=0.$$

Theorem 4. Let the assumptions of theorem 3 hold. If exists a function $\phi(v)=L^1(Ra)$ such that:

$$(37) \qquad f(v,t)\le\phi(v,t) \qquad \text{a.e.}$$

then the steady solution to equation (2) is unconditionally attractive in the L^1-norm.

Proof. Because

$$\lim_{t\to\infty} f(v,t)=M(v) \qquad \text{a.e.}$$

by the hypothesis of theorem it follows that $M(v)\in L^1(Ra)$ and:

$$(38) \qquad \lim_{t\to\infty} \int_{\mathbb{R}a} |f(v,t)-M(v)|dv=0$$

then

$$(39) \qquad \lim_{t\to\infty} \| u(v,t) \|=0$$

and the theorem is proved.

5. L^1-Asymptotic Stability.

We start recalling the definitions of continuous dependence and stabil i ty in the L^1-norm according to Liapunov.

Definition 2. The solution f is said to depend continuously on the initial data, iff

$$(40) \quad \forall T,\varepsilon>0 \ \ \exists\ \delta(\varepsilon,T)>0 : \ \|u_o\|<\delta \ \Rightarrow\ \|u(v,t)\|<\varepsilon \ \ \forall t\in[0,T]$$

Definition 3. The solution f is said to be stable, with respect to the initial data iff

$$(41)\ \forall\varepsilon>0\ \exists\ \delta(\varepsilon)\ :\ \|u_0\|<\varepsilon\quad \|u(v,t)\|<\varepsilon\quad \forall\ t\in[0,\infty)$$

Definition 4. The solution f is called asymptotically stable if it is both stable and attractive.

Theorem 5. Suppose that assumptions of theorem 4 are verified. Then the solution M(v) to equation (2) is uniformly asymptotically stable in L^1-norm.

Proof. Since the steady solution is unconditionally L^1-attractive we have:

$$\text{(i)}\qquad \forall\ \varepsilon>0\ \exists\ T>0\ :\ t>T\ \Rightarrow\ \|u(v,t)\|<\varepsilon\ .$$

Moreover the solutions to equation (2) depend continuously on the initial data in the L^1-norm [7] i.e.:

$$\text{(ii)}\quad \forall\ \varepsilon>0, T>0,\ \exists\ \delta>0\ :\ \|u_0\|<\delta\ \Rightarrow\ \|u(v,t)\|<\varepsilon\ \ t\in[0,\infty)$$

by (i) and (ii) (41) follows. Therefore accounting for theorem 4, theorem 5 immediately follows.

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REFERENCES

[1] Boffi V.C.,Molinari V.G.,Nonlinear Transport Problems by Factorization of the Scattering Probability, Il Nuovo Cimento Vol. 65 B, N.1, p. 29-44 (1981).

[2] Boffi V.C.,Spiga G.,Rigorous iterated solutions to a nonlinear integral evolution problem in particle transport

theory, J. Math. Phys., **23**, 1859,1982, (1982).
[3] Boffi V.C.,Spiga G.,Global solution to a nonlinear integral evolution problem in particle transport theory, J. Math. Phys., **23**, 2299,2303, (1982).
[4] Cercignani C.,The Boltzmann Equation and Its Applications, Springer Verlag New York Inc. (1988).
[5] Cercignani C., Lampis M., Sgarra C., On the relation between the scattering kernel and the standard formulation of the Boltzmann equation, Il Nuovo Cimento, Vol.101 B, N.5 p.523-531,(1988).
[6] Rionero S.,Global solutions to a nonlinear integral evolution problem in particle transport theory, Atti Accademia Scienze Torino, Suppl. Vol. 120 p.199-209,(1986).
[7] Rionero S.,Guerriero G.,On the stability of the solutions to a nonlinear integral evolution problem arising in particle transport theory, Meccanica **22**,p.59-64, (1987).
[8] Rionero S.,Guerriero G.,L^1-stability of the solutions to an integral evolution equation of the nonlinear particle transport theory, Meccanica,in press.
[9] Spiga G.,In Proceedings of the 1st German Italian Symposium on the Applications of the Mathematics in Industry and Technology Boffi V.C.,Neunzert H. Eds.p.430, Teubner,Stuttgart, pp. 431-448, (1984).
[10] Spiga G.,Nonnenmacher T.,Boffi V.C.,Moment equations for the diffusion of the particles of a mixture via the scattering kernel formulation of the nonlinear Boltzmann equation,Phisica 131A, p.431-448, (1985).
[11] Waldmann L.,Transporterscheinungen in Gasen von mittlerem Druck,Handbuck der Physik,Vol.12, Ed. S. Flugge Springer Berlin (1958).

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